**GOVERNMENT ARTS AND SCIENCE COLLEG, KOVILPATTI – 628 503.**

(AFFILIATED TO MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI)

DEPARTMENT OF MATHEMATICS

STUDY E - MATERIAL

CLASS : I M.SC (MATHEMATICS) SEM: I

SUBJECT : ORDINARY DIFFERENTIAL EQUATIONS(PMAM15)

## 1.4 Paper 4: ORDINARY DIFFERENTIAL EQUATIONS

**Text Book:** Differential Equations with application and historical notes, G.F. Simmons, Second Edition, Tata McGraw Hill.

**Unit I:** **Second Order linear equations :** General solution of the Homogeneous equations – The use of a known solution to find another – The method of variation of parameters.
**Sections:** 14 – 16.

**Unit II:** **Power series solutions:** A review of power series solutions – Series solution of first order equations – Second order equations – Ordinary points.
**Sections:** 26 – 28.

**Unit III:** Regular singular points – Legendre polynomials- Properties of Legendre polynomials
**Sections:** 29, 30, 44, 45.

**Unit IV:** Bessel functions – The Gamma functions – Properties of Bessel functions.
**Sections:** 46, 47.

**Unit V:** **Linear systems :** Homogeneous linear systems with constant coefficients
**Sections:** 55, 56.

paper IV

# ORDINARY DIFFERENTIAL EQUATIONS

## Unit - I

General solution of the homogenous equation - the use of a known solution to find another - The method of variation of Parameters.

Section : 14-16

## Unit - II

Power Series Solution: A review of Power series solution of first order equations - second order equations - Ordinary Points.

Section: 26 - 28

## Unit - III

Regular Singular Points - Legendre Polynomial - Properties of legendre polynomial

Section :- 29, 30, 44, 45

## Unit - IV

Bassel function - The Gamma function - Properties of Based function

Section : 46, 47

Unit - V

linear System: Homogenous linear Systems with constant coefficient

Section : 55, 56

Text book:

Differential equations with application and historical notes. by G.F. Simmons.

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# UNIT - I

## First order differential equations:-

$$\frac{dy}{dx} + P(x)y = Q(x)$$ where

$P(x)$ and $Q(x)$ are functions of $x$

general solution first order differential equation.

$$\boxed{y e^{\int p\,dx} = \int Q e^{\int p\,dx}\,dx + c}$$

where $c$ is a constant.

$y(I.F) = \int Q(I.F)\,dx + c$

$I.F = e^{\int p\,dx}$

$a\left(\frac{dy}{dx}\right) + b\left(\frac{dy}{dx}\right)^2 = 2 \rightarrow$ first order

## Second order differential equation:-

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad \text{—— (1)}$$

where $P(x)$, $Q(x)$ and $R(x)$ are functions of $x$

**Note:**

If $R(x) = 0$, then the equation (1) is said to be homogenous second order differential equation

If $R(x) \neq 0$, then the equation (1) is said to be non-homogeneous second order differential equation

Formation of an differential equation by eliminating $c_1$ and $c_2$:-

① $y = c_1 x + c_2 x^2$ —— ①

differenciate ① w.r to $x$

$$\frac{dy}{dx} = c_1 + 2c_2 x \quad \text{—— ②}$$

Diff ② w.r.to $x$

$$\boxed{\frac{d^2y}{dx^2} = 2c_2}$$

$$\therefore c_2 = \frac{1}{2}\frac{d^2y}{dx^2} = \frac{y''}{2}$$

$$② \Rightarrow y' = c_1 + y''(x)$$

$$\therefore c_1 = y' - y''x$$

$$① \Rightarrow y = (y' - y''x)x + \frac{1}{2}y''x^2$$

$$y = y'x - y''x^2 + \frac{1}{2}y''x^2$$

$$y = xy' - x^2y'' + \frac{1}{2}x^2y''$$

$$y = xy' - \frac{1}{2}x^2y''$$

$$2y = 2xy' - x^2y''$$

$$x^2y'' - 2xy' + 2y = 0 \,//.$$

(2) $y = c_1 e^{kx} + c_2 e^{-kx}$ —— (1)

Diff (1) w.r.to $x$

$y' = c_1 e^{kx} k + c_2 e^{-kx}(-k)$ —— (2)

Diff (2) w.r.to $x$

$y'' = c_1 e^{kx} k^2 + c_2 e^{-kx}(k^2)$

$y'' = k^2 (c_1 e^{kx} + c_2 e^{-kx})$

$y'' = k^2 y$ (∴ by eqn (1))

$\therefore y'' - k^2 y = 0$ //.

(3) $y = c_1 \sin kx + c_2 \cos kx$

soln:

$y = c_1 \sin kx + c_2 \cos kx$ —— (1)

Diff (1) w.r.to $x$

$y' = c_1 \cos kx . k - c_2 \sin kx . k$ —— (2)

Diff (2) w.r.to $x$

$y'' = -c_1 \sin kx \, k^2 - c_2 \cos kx \, k^2$

$= -k^2 [c_1 \sin kx + c_2 \cos kx]$

$y'' = -k^2 y$

$\therefore y'' + k^2 y = 0$ //.

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(4). $y = c_1 + c_2 e^{-2x}$

Soln:

$$y = c_1 + c_2 e^{-2x} \quad \text{——— (1)}$$

Diff (1) w.r.to $x$

$$y' = c_2 e^{-2x} (-2) \quad \text{——— (2)}$$

Diff (2) w.r.to $x$

$$y'' = c_2 e^{-2x} 4 \quad \text{——— (3)}$$

From (2) and (3), we get

$$y'' = 2 \left(2 c_2 e^{-2x}\right)$$

$$= 2(-y')$$

$$y'' + 2y' = 0$$ //.

(5). $y = c_1 x + c_2 \sin x$

Soln:

$$y = c_1 x + c_2 \sin x \quad \text{——— (1)}$$

Diff (1) w.r. to $x$

$$y' = c_1 + c_2 \cos x \quad \text{——— (2)}$$

Diff (2) w.r.to $x$

$$y'' = -c_2 \sin x$$

$\therefore \quad c_2 = -\dfrac{y''}{\sin x}$

$(2) \Rightarrow \quad y' = c_1 - \left(\dfrac{y''}{\sin x}\right)\cos x$

$y' = c_1 - \cot x \, y''$

$c_1 = y' + \cot x \, y''$

$(1) \Rightarrow \quad y = (y' + \cot x \, y'')x - \left(\dfrac{y''}{\sin x}\right)\sin x$

$y = xy' + x \cot x \, y'' - y''$

$y'' - x\cot x \, y'' - xy' + y = 0$

$(1 - x\cot x)\, y'' - xy' + y = 0$ //.

(6) $y = c_1 e^{x} + c_2 e^{-3x}$

Soln:

$y = c_1 e^{x} + c_2 e^{-3x}$ ——— (1)

Diff (1) w.r.to $x$

$y' = c_1 e^{x} + c_2 e^{-3x}(-3)$ ——— (2)

Diff (2) w.r.to $x$

$y'' = c_1 e^{x} + c_2 e^{-3x} \cdot 9$ ——— (3)

$(3) - (2) \Rightarrow y'' - y' = 12 c_2 e^{-3x}$

$$\therefore C_2 = \frac{y'' - y'}{12e^{-3x}}$$

③ $\Rightarrow$ $$y' = C_1 e^{x} - 3\left(\frac{y''-y'}{12e^{-3x}}\right)e^{-3x}$$

$$y' = c_1 e^{x} - \frac{1}{4}(y''-y')$$

$$\therefore \quad C_1 = \frac{y' + \frac{1}{4}(y''-y')}{e^{x}}$$

① $\Rightarrow$

$$y = \frac{\left(y' + \frac{y''}{4} - \frac{y'}{4}\right)}{e^{x}} e^{x} + \left(\frac{y''-y'}{12e^{-3x}}\right)e^{-3x}$$

$$= y' + \frac{y''}{4} - \frac{y'}{4} + \frac{y''-y'}{12}$$

$$12y = 12y' + 3y'' - 3y' + y'' - y'$$

$\cancel{4y'' - 9}$.

$$12y = 4y'' + 8y'$$

$$4y'' + 8y' - 12y = 0$$

$$y'' + 2y' - 3y = 0$$

//.

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**Theorem: (A) (uniquness theorem)**

Let $P(x)$, $Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a,b]$. If $x_0$ is any point in the interval $[a,b]$, and if $y_0$ and $y_0'$ are any numbers whatever, then the second order differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

has one and only one solution $y(x)$ in the entire closed, interval $[a,b]$ such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$

**Problem**

(1) Find the solution of the initial variate value Problem $y''+y=0$, $y(0)=0$ and $y'(0)=1$

$$y''+y=0 \quad \text{—— (1)}$$

**Soln:**

we know that $y=\sin x$, $y=\cos x$ and $y = c_1\sin x + c_2\cos x$ are all the solutions of (1) —— (2)

$y(0) = c_1 \sin(0) + c_2 \cos(0)$

$0 = 0 + c_2$

$\boxed{\therefore c_2 = 0}$

$y' = c_1 \cos x - c_2 \sin x$

$y'(0) = c_1 \cos x - c_2 \sin x$

$y'(0) = c_1 \cos(0) - c_2 \sin(0)$

$1 = c_1 - 0$

$\boxed{\therefore c_1 = 1}$

$\therefore$ (2) becomes

$y = 1 \cdot \sin x + (0) \cos x$

$\boxed{y = \sin x}$

$\therefore y = \sin x$ is the only solution of the second order differential equation $y'' + y = 0$

(2). Find the Solutions of the initial value Problem $y'' + y = 0$, $y(0) = 1$ and $y'(0) = 0$.

Soln:

$y'' + y = 0$ ——— (1)

we know that $y = \sin x$,

$y = \cos x$ and $y = c_1 \sin x + c_2 \cos x$

are all the solutions of (1) ——— (2)

$y(0) = c_1 \sin(0) + c_2 \cos(0)$

$1 = 0 + c_2$

$\boxed{\therefore c_2 = 1}$

$y' = c_1 \cos x - c_2 \sin x$

$y'(0) = c_1 \cos(0) - c_2 \sin(0)$

$0 = c_1 - 0$

$\boxed{\therefore c_1 = 0}$

$\therefore$ (2) becomes

$y = 0 \cdot \sin x + (1) \cos x$

$\boxed{y = \cos x}$

$\therefore y = \cos x$ is the only solutions of the second order differential equation.

Note:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + \partial(x)y = R(x) \quad \text{---} (1)$$

Now it is reduced to

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + \partial(x)y = 0 \quad \text{---} (2)$$

The eqn (1) is called the complete equations and (2) is called the reduced equation associated ~~~~ with it

Theorem: (B)

If $y_g$ is the general soln of $y'' + p(x)y' + Q(x)y = 0$ and $y_p$ is any particular solution of $y'' + p(x)y' + Q(x)y = R(x)$, then $y_g + y_p$ is the general solution of $y'' + p(x)y' + Q(x)y = R(x)$

Proof: Consider,

$$y'' + p(x)y' + Q(x)y = R(x) \quad \text{---} (1) \quad \text{and}$$

$$y'' + p(x)y' + Q(x)y = 0 \quad \text{---} (2)$$

Given $y_g$ in soln of (2)

$$y''_g + p(x) y_g' + Q(x) y_g = 0 \quad \text{—— (3)}$$

Also given $y_p$ is solution of (1)

$$y_p'' + p(x) y_p' + Q(x) y_p = R(x) \quad \text{—— (4)}$$

claim

$y_g + y_p$ in a soln of (1)

(i.e) $(y_g + y_p)'' + p(x)(y_g + y_p)' + Q(x)(y_g + y_p)$

$= R(x)$

consider,

$$(y_g + y_p)'' + p(x)(y_g + y_p)' + Q(x)(y_g + y_p)$$

$$= y_g'' + y_p'' + p(x) y_g' + p(x) y_p' + Q(x) y_g + Q(x) y_p$$

$$= (y_g'' + p(x) y_g' + Q(x) y_g) + (y_p'' + p(x) y_p' + Q(x) y_p)$$

$= 0 + R(x)$ (by (3) and (4))

$= R(x)$

$\therefore$ $y_g + y_p$ in a solution of (1)

Theorem : (C) (linear combination)

If $y_1(x)$ and $y_2(x)$ are any two solutions of $y'' + P(x)y' + Q(x)y = 0$ then, $c_1y_1(x) + c_2y_2(x)$ is also a solution $y'' + P(x)y' + Q(x)y = 0$, for any constants $c_1$ and $c_2$

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**Proof-**

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{——— (1)}$$

Since, $y_1(x)$ is a solution of (1)

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \quad \text{——— (2)}$$

Since, $y_2(x)$ is a solution of (1),

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0 \quad \text{——— (3)}$$

claim.

$y = c_1y_1(x) + c_2y_2(x)$ is a solution of (1).

consider,

$$(c_1y_1 + c_2y_2)'' + P(x)(c_1y_1 + c_2y_2)'$$
$$+ Q(x)(c_1y_1 + c_2y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + P(x)\, c_1 y_1' + P(x)\, c_2 y_2' + Q(x)\, c_1 y_1 + Q(x)\, c_2 y_2$$

$$= c_1 \left( y_1'' + P(x) y_1' + Q(x) y_1 \right) + c_2 \left( y_2'' + P(x) y_2' + Q(x) y_2 \right)$$

$$= 0 + 0 = 0$$

Hence $c_1 y_1 + c_2 y_2$ is also a solution of (1).

Note:

The above theorem can be restated as any linear combination of two solutions of homogenous equation is also a solution of the homogenous equation.

Defn:

Two functions $f(x)$ and $g(x)$ defined on the interval $[a,b]$ are said to be linearly dependent if one is constant multiple of other otherwise they are linearly independent.

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**Note:**

If one of the function is identically zero, then they are linearly independent.

**Defn:** (wronskian)

One word

wronskian of $y_1$ and $y_2 = W(y_1, y_2)$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

**Theorem:**

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{--- (1)}$$

on the interval $[a, b]$. Then

$$c_1 y_1(x) + c_2 y_2(x) \quad \text{--- (2)}$$

is the general solution of (1) on $[a, b]$ in the sense that every solution of (1) on this interval can be obtained from (2) by

Suitable choice of one arbitary constants $c_1$ and $c_2$.

First we need to prove the following Lemmas.

**Lemma: (i)**

If $y_1$ and $y_2$ are any two solutions of (1) on $[a,b]$ then their wronskian $w = W(y_1, y_2)$ is either <u>identically zero or never zero</u> on $[a,b]$

linearly independent

**Proof:**

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

$$w' = y_1 y_2'' + \cancel{y_2' y_1'} - y_2 y_1'' - \cancel{y_1' y_2'}$$

$$= y_1 y_2'' - y_2 y_1''$$

Since $y_1$ and $y_2$ are solutions of (1),

$$y_1'' + P(x) y_1' + Q(x) y_1 = 0 \quad \text{——— (3)}$$

$$y_2'' + p(x)\, y_2' + Q(x)\, y_2 = 0 \quad \text{--- (4)}$$

(3) $\times y_2$ ; $\quad y_1'' y_2 + p(x)\, y_1' y_2 + Q(x)\, y_1 y_2 = 0$ --- (5)

(4) $\times y_1$ ;- $\quad y_1 y_2'' + p(x)\, y_2' y_1 + Q(x)\, y_2 y_1 = 0$ --- (6)

(5) - (6)

$$y_1'' y_2 - y_1 y_2'' + (y_1' y_2 - y_1 y_2')\, p(x) = 0$$

$$-w' - w\, p(x) = 0$$

$\boxed{w' = \frac{dw}{dx}}$

$$\Rightarrow \quad w' = -w\, p(x)$$

$$\Rightarrow \quad \frac{dw}{dx} = -p(x)\, w$$

$$\Rightarrow \quad \frac{dw}{w} = -p(x)\, dx$$

$$\Rightarrow \int \frac{dw}{w} = \int -p(x)\, dx$$

$$\Rightarrow \log w = -\int p(x)\, dx + c$$

$$\Rightarrow w = e^{-\int p(x)\, dx + c}$$

$$= e^{-\int p(x)\, dx} \cdot e^{c}$$

$$w = k \cdot e^{-\int p(x)\, dx}$$

where, $k = e^c$

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Since the exponential factor is never zero,

if the constant $k=0$, then $w$ is zero,

if the constant $k\neq 0$, then $w$ is never zero.

Lemma: 2

If $y_1(x)$ and $y_2(x)$ are two solutions of equation (i) on $[a,b]$, then they are linearly dependent on this interval iff the wronskian of $y_1$ and $y_2$ is identically zero.

Proof:

Assume that $y_1$ and $y_2$ are linearly dependent

Now, $$w(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

without loss of generality

If either function $y_1$ and $y_2$ are identically zero, then the conclusion is clear.

W.L.G. Assume that neither is identically zero

Since $y_1$ and $y_2$ are linearly independent,

$$y_2 = ky_1 \quad \text{—— (1)}$$

$$y_2' = ky_1' \quad \text{—— (2)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{y_2'}{y_2} = \frac{\cancel{k}y_1'}{\cancel{k}y_1}$$

$$y_1 y_2' = y_1' y_2$$

$$\Rightarrow y_1 y_2' - y_1' y_2 = 0$$

$$\Rightarrow W = 0$$

Conversely, Assume that wronskian of $y_1$ and $y_2$ is identically zero.

To Prove:

$y_1$ and $y_2$ are linearly dependent. If $y_1$ is identically zero on [a,b], then the functions are linearly dependent

$\therefore$ we assume that $y_1$ does not vansih identically on [a,b]

Now, $W = 0$

$$\Rightarrow y_1 y_2' - y_1' y_2 = 0$$

$$\Rightarrow \frac{y_1 y_2' - y_1' y_2}{y_1^2} = 0$$

$$\Rightarrow d(y_2/y_1) = 0$$

$$\Rightarrow \frac{y_2}{y_1} = k, \text{ where } k \text{ is constant}$$

$$\Rightarrow y_2 = k y_1$$

$\therefore y_1$ and $y_2$ are linearly dependent

Proof of the main theorem:-

Let $y(x)$ be any solution of (1) on [a,b].

We must show that one can find constants $c_1$ and $c_2$ such that

$y(x) = c_1 y_1(x) + c_2 y_2(x)$ on $[a,b]$

By existence and uniqueness theorem it is enough to show that for any point $x_0 \in [a,b]$, we can find $c_1$ and $c_2$ such that

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0) \text{ and}$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$$

For this system to be solvable for $c_1$ and $c_2$, it suffices that

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0) y_2'(x_0) - y_2(x_0) y_1'(x_0)$$

have a value different from zero

i.e) By lemma, there exists $x_0$ in $[a,b]$ such that $y_1(x_0) y_2'(x_0) - y_2(x_0) y_1'(x_0)$ is non-zero

Hence the proof.

Problem

Show that $y = c_1 \sin x + c_2 \cos x$ is general solution $y'' + y = 0$ on any interval. Find particular solutions for which $y(0) = 2$ and $y'(0) = 3$.

Soln:

Given, $y'' + y = 0$ ——— (1)

Let $y_1 = \sin x$

$y_1' = \cos x$ and $y_1'' = -\sin x$

(1) $\Rightarrow$ $y_1'' + y = -\sin x + \sin x = 0$

Hence $y_1 = \sin x$ is the Solution of (1)

Let $y_2 = \cos x$

$y_2' = -\sin x$ and $y_2'' = -\cos x$

(1) $\Rightarrow$ $y_2'' + y = \cos x - \cos x = 0$

Hence $y_2 = \cos x$ is a soln of (1)

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\cos^2 x + \sin^2 x)$$

$$= -1 \neq 0.$$

Hence $y_1$ and $y_2$ are linearly independent.

Comparing eqn (1) with the generally equation,

$y'' + P(x) y' + Q(x) y = 0$, we get

$P(x) = 0$ and $Q(x) = 1$

Hence $P(x)$ and $Q(x)$ are constant functions.

$\therefore$ $P(x)$ and $Q(x)$ are continuous functions

$\therefore$ By thm,

$y = c_1 \sin x + c_2 \cos x$ is the general solution of (1)

$\cancel{y(x)}\ y = c_1 \sin \cancel{x} + c_2 \cos x$

$\cancel{y(0) = c_1}$

$y(0) = c_1 \sin 0 + c_2 \cos 0$

$2 = 0 + c_2$

$\therefore c_2 = 2$

$y_1' = c_1 \cos x - c_2 \sin x$

$y'(0) = c_1 \cos 0 - c_2 \sin 0$

$3 = c_1 - 0$

$\boxed{\therefore c_1 = 3}$

$\therefore$ Particular soln is

$y = 3 \sin x + 2 \cos x$ //.

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(2) show that $e^x$ and $e^{-x}$ are linearly independent soln of $y'' - y = 0$ on any interval.

Soln:

Given $y'' - y = 0$ ——— (1)

let $y_1 = e^x$ ; $y_2 = e^{-x}$

$y_1$ is the solution of equation (1)

$y_1' = e^x$ and $y_1'' = e^x$

(i) $\Rightarrow$ $e^x - e^x = 0$

Hence $y_1$ is the soln of the equation.

(i) $\Rightarrow$ $y_2 = e^{-x}$ ; $y_2' = -e^{-x}$

and $y_2'' = e^{-x}$

(i) $\Rightarrow$ $e^{-x} - e^{-x} = 0$

Hence $y_2$ is the solution of the equation (i)

To Prove $y_1$ and $y_2$ are linearly independent

It is enough to Prove that

$w(y_1, y_2) \neq 0$.

$$w(y_1, y_2) = w(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -e^x e^{-x} - e^x e^{-x}$$

$$= -2e^x e^{-x}$$

$$= -2e^0$$

$w(y_1, y_2) = -2 \neq 0.$

Hence $y_1$ and $y_2$ are linearly independent soln of the equation ① on any interval.

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③ show that $y_1 = C_1 x + C_2 x^2$ is the general solution of

$x^2 y'' - 2xy' + 2y = 0$ on any interval not containing zero and find the Particular solution for $y(1) = 3$ and $y'(1) = 5$

Soln:

Given $x^2 y'' - 2xy' + 2y = 0$ ——— ①

Let $y_1 = x$ and $y_2 = x^2$

$y' = 1$ and $y'' = 0$

① $\Rightarrow x^2(0) - 2x(1) + 2x = 0$

Hence $y_1 = x$ is the soln eqn ①

Let $y_2 = x^2$

$y_2' = 2x$ and $y_2'' = 2$

① $\Rightarrow = x^2(2) - 2(x)(2x) + 2(x^2)$

$\Rightarrow 2x^2 - 4x^2 + 2x^2 = 0$

$\Rightarrow 4x^2 - 4x^2 = 0$

Hence $y_2 = x^2$ is the soln of ①

$$w(y_1, y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2 \neq 0$$

Hence $y_1$ and $y_2$ are linearly independent.

Comparing the eqn ① with the general equation

$y'' + P(x) y' + Q(x) y' = 0$, we get

$P(x) = \frac{-2x}{x^2} = 2x$ and $Q(x) = \frac{2}{x^2}$

Hence, $P(x)$ and $Q(x)$ are function of $x$ also polynomial function are continuous

By theorem

$y = c_1 x + c_2 x^2$ is the general solution of eqn ①

$y(1) = c_1(1) + c_2(1)$

$3 = c_1 + c_2$ —— (2)

$y' = c_1 + 2c_2 x$

$y'(1) = c_1 + 2c_2$

$5 = c_1 + 2c_2$ —— (3)

(3) - (2) $\boxed{c_2 = 2}$

(2) => $c_1 + c_2 = 3$

$c_1 + 2 = 3$

$\boxed{c_1 = 1}$

$y = x + 2x^2$ is a Particular theorem

4. Show that $y = c_1 e^{x} + c_2 e^{2x}$ is the general soln of $y'' - 3y' + 2y = 0$ on any interval find the Particular soln for which $y(0) = -1$ and $y'(0) = 1$.

Soln:-

Given $y'' - 3y' + 2y = 0$ —— (1)

Let $y = c_1 e^{x} + c_2 e^{2x}$

and let

$y_1 = e^x$ | $y_2 = e^{2x}$

$y_1' = e^x$ | $y_2' = 2e^{2x}$

$y_1'' = e^x$ | $y_2'' = 4e^{2x}$

(1) $\Rightarrow y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x$

$y_1 = e^x$ is a solution of eqn (1)

(1) $\Rightarrow y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 3(2e^{2x})$
$+ 2e^{2x} = 0.$

$y_2 = e^{2x}$ is a solution of eqn (1)

~~(1) $\Rightarrow y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 3(2e^{2x}$~~

dependent $=0$
indep $\neq 0$

To Prove
$y$ is linearly independent

Enough to prove that $\boxed{W(y_1, y_2) = 0}$

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$$= (e^x)(2e^{2x}) - e^x \cdot e^{2x}$$

$$= e^{3x} \neq 0.$$

$2e^x e^{2x} - e^x \cdot e^{2x}$

$2e^{3x} \quad e^{3x}$

$\therefore y$ is linearly independent

Comparing ① with

$y'' + P(x)y' + Q(x)y = 0$;

$y'' - 3y' + 2y = 0$

$P(x) = -3$ and $Q(x) = 2$

$y'' + p(x)dy + Q(x)y =$

Hence $P(x)$ and $Q(x)$ are Constant function.

$P(x)$ and $Q(x)$ are Continuously in $[a,b]$.

$e^{2x}$

$2e^{2x}$

By theorem,

$y = c_1 e^x + c_2 e^{2x}$ is a general solution of the equation

$y(0) = c_1 e^0 + c_2 e^0$

$-1 = c_1 + c_2 \Rightarrow$ ③

$c_1 + c_2 = -1$

$y'(x) = c_1 e^x + 2c_2 e^{2x}$

$y'(0) = c_1 e^0 + 2c_2 e^{2(0)}$

$1 = c_1 + 2c_2$ —— ④

$c_1 + 2c_2 = 1$

④ - ③

$\Rightarrow \boxed{c_2 = 2}$

$\boxed{c_1 = -3}$

④ - ③

$c_1 + 2c_2 = 1$

$c_1 + c_2 = -1$

$c_2 = 2$

$y = -3e^x + 2e^{2x}$ is a Particular solution of the equation ①

Show that $y = c_1 e^{2x} + c_2 x e^{2x}$ is the general solution of $y'' - 4y' + 4y = 0$ on any interval

Soln:

Given $y'' - 4y' + 4y = 0$ —— (1)

$$y = C_1 e^{2x} + C_2 x e^{2x}$$

$y_1 = e^{2x}$

$y_1' = 2e^{2x}$

$y_1'' = 4e^{2x}$

$y_2 = x e^{2x}$

$y_2' = 2xe^{2x} + e^{2x}$

$y_2'' = 4xe^{2x} + 2e^{2x} + 2e^{2x}$

$y_2'' = 4x e^{2x} + 4e^{2x}$

$y_1'' - 4y_1' + 4y = 4e^{2x} - 4(2e^{2x}) + 4e^{2x} = 0$

$8e^{2x} - 8e^{2x} = 0$

$y_1$ is the solution of eqn (1)

$y_2'' - 4y_2' + 4y_2$

(1) $\Rightarrow 4xe^{2x} + 4e^{2x} - 4(2xe^{2x} + e^{2x}) + 4xe^{2x}$

$= 4xe^{2x} + 4e^{2x} - 8xe^{2x} - 4e^{2x} + 4xe^{2x}$

$= 0$

Hence $y_2$ is the soln of the equation (1)

$$w(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix}$$

$$= (e^{2x})(2xe^{2x} + e^{2x}) - 2e^{2x} \cdot xe^{2x}$$

$2xe^{4x} + e^{4x}$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x} \neq 0.$$

$y_1$ and $y_2$ are linearly independent

comparing (1) with equation

$y'' + P(x)y' + Q(x)y = 0$, we get

$P(x) = -4$ and $Q(x) = 4$.

$\therefore$ $P(x)$ and $Q(x)$ are Constant functions.

Hence $P(x)$ and $Q(x)$ is a general solution for (1).

---

(6) By inspection or experiment, find two linearly independent soln of $x^2y'' - 2y = 0$ —— (1) on the interval $[1,2]$ and the determine the Particular solution satisfying the initial conditions $y(1) = 1$; $y'(1) = 8$

Soln:

let $y_1 = x^2$

$y_1' = 2x$ and

$y_1'' = 2$

From ① $\Rightarrow x^2 y_1'' - 2y_1 = 0$

$x^2 (2) - 2(x^2) = 2x^2 - 2x^2 = 0$

Hence $y_1$ is a solution of ①

Let $y_2 = \frac{1}{x}$

$y_2' = \frac{-1}{x^2}$ and $y_2'' = \frac{2}{x^3}$

Now $x^2 y_2'' - 2y_2 = x^2 \left(\frac{2}{x^3}\right) - 2\left(\frac{1}{x}\right) = 0$

Hence $y_2$ is a solution of ①

$w(y_1, y_2) \neq 0$.

Hence $y_1$ and $y_2$ are linearly independent solution of ①.

Comparing ① with the general second order differential equation

$y'' + P(x) y' + Q(x) y = 0$

$P(x) = 0$ and $Q(x) = \frac{-2}{x^2}$

$P(x)$ and $Q(x)$ are continuously functions By theorem

$y = c_1 x^2 + c_2 \left(\frac{1}{x}\right)$ is a general solution of equation ①.

Now, $y(1) = c_1(1) + c_2(1)$

$1 = c_1 + c_2$

$\therefore c_1 + c_2 = 1$ ——— ②

$y'(1) = 2x c_1 - \frac{1}{x^2} c_2$

$2c_1 - c_2 = 8$ ——— ③

② + ③ $\Rightarrow$ $3c_1 = 9$

$\boxed{c_1 = 3}$

② $\Rightarrow$ $3 + c_2 = 1$

$\boxed{c_2 = -2}$

$y = 3x^2 - \frac{2}{x}$ is the Particular solution of the equation ①.

---

Pb: ⑪ In each of the following, verify the function $y_1(x)$ and $y_2(x)$ are linearly independent solution of the given differential equation on the interval $[0,2]$ and find the solution satisfying the initial conditions.

~~Soln:~~ (a.) $y''+y'-2y=0$, $y_1=e^{x}$ and $y_2=e^{-2x}$, $y(0)=8$; and $y'(0)=2$

(b). $y''+y'-2y=0$, $y_1=e^{x}$ and $y_2=e^{-2x}$, $y(1)=0$ and $y'(1)=0$.

(c.) $y''+5y'+6y=0$, $y_1=e^{-2x}$, $y_2=e^{-2x}$, $y(0)=1$ and $y'(0)=1$.

(d.) $y''+y'=0$; $y_1=1$; $y_2=e^{-x}$ $y(2)=0$; $y'(2)=e^{-2}$

Soln:

(a). Given $y''+y'-2y=0$ ——— (1)

Let $y_1=e^{x}$ and $y_2=e^{-2x}$

| | |
|---|---|
| $y_1'=e^{x}$ | $y_2'=-2e^{-2x}$ |
| $y_1''=e^{x}$ | $y_2''=4e^{-2x}$ |

(1) $\Rightarrow$ $y_1''+y_1'-2y_1=0$

$\Rightarrow e^{x}+e^{x}-2e^{x}=0$

Hence $y_1$ is the solution of (1)

(1) $\Rightarrow y_2''+y_2'-2y=4e^{-2x}-2e^{-2x}-2e^{-2x}$

$=4e^{-2x}-4e^{-2x}=0$

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Hence $y_2$ is the soln of ①

$$w(y_1, y_2) = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^x \cdot (-2) e^{-2x} - e^x \cdot e^{-2x}$$

$$= -2e^{-x} - e^{-x}$$

$$= -3e^{-x} \neq 0 ..$$

Hence $y_1$ and $y_2$ are linearly indepent

Comparing ① with second order differential equation

$P(x) = 1$ and $Q(x) = -2$

Hence $P(x)$ and $Q(x)$ are Continuous function

By theorem,

$y = c_1 \cdot e^x + c_2 e^{-2x}$ is a general soln of equation ①.

Now $y(0) = c_1 e^0 + c_2 e^0$

$$\boxed{c_1 + c_2 = 8} \text{ --- } ②$$

$$y'(0) = c_1 e^x - 2c_2 e^{-2x}$$

$$2 = c_1 e^0 - 2c_2 e^0$$

$$\therefore c_1 - 2c_2 = 2 \text{ --- } ③$$

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② - ③

$\Rightarrow \quad 6 = C_2 + 2C_2$

$6 = 3C_2$

$\boxed{C_2 = 2}$

② $\Rightarrow \boxed{C_1 = 6}$

$y = 6e^{x} + 2e^{-2x}$ is the Particular solution of the equation ①

ⓑ. $y = c_1 e^{x} + c_2 e^{-2x}$ is the general soloution of the equation ①

Now $y(1) = c_1 e^{1} + c_2 e^{-2}$

$0 = c.e^{1} + c_2 e^{-2}$ —— ②

$y'(1) = c.e^{x} - 2c_2 e^{-2x}$

$0 = c_1 e^{1} - 2c_2 e^{-2}$ —— ③

② - ③

$\Rightarrow \quad c_2 e^{-2} + 2c_2 e^{-2}$

$0 = 3c_2 e^{-2}$

$\therefore \boxed{c_2 = 0}$

② $\Rightarrow \quad c_1 e^{1} + 0 = 0$

$c_1 = 0$

$y = 0$ is the Particular solution of the equation ①.

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(8). Given

$$y'' + 5y' + 6y = 0 \quad \text{——— (1)}$$

$y_1 = e^{-2x}$ and $y_2 = e^{-3x}$

$y_1' = -2e^{-2x}$ | $y_2' = -3e^{-3x}$

$y_1'' = 4e^{-2x}$ | $y_2'' = 9e^{-3x}$

(1) $\Rightarrow y_1'' + 5y_1' + 6y_1 = 0$

$\Rightarrow 4e^{-2x} + 5(-2e^{-2x}) + 6(e^{-2x}) = 0$

$\Rightarrow 4e^{-2x} - 10e^{-2x} + 6e^{-2x} = 0$

Hence $y_1 = e^{-2x}$ is the soln of (1)

(1) $\Rightarrow y_2'' + 5y_2' + 6y_2 = 0$

$\Rightarrow 9e^{-3x} + 5(-3e^{-3x}) + 6(e^{-3x})$

$= 9e^{-3x} - 15e^{-3x} + 6e^{-3x}$

$= 0$

Hence $y_2 = e^{-3x}$ is the soln of the equation (1)

$$w(y_1, y_2) = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = (e^{-2x})(-3e^{-3x}) - (e^{-3x})(-2e^{-2x})$$

$$= -3e^{-5x} + 2e^{-5x} = e^{-5x} \neq 0.$$

Hence $y_1$ and $y_2$ are linearly independent.

Comparing (1) with second order differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

$P(x) = 5$ ; $Q(x) = 6$.

$P(x)$ and $Q(x)$ are continuous function

By theorem,

$y = c_1 e^{-2x} + c_2 e^{-3x}$ is a general solution for (1)

$$y(0) = c_1 e^0 + c_2 e^0$$

$$c_1 + c_2 = 1 \quad \text{——— (2)}$$

$$y' = -2c_1 e^{-2x} - 3c_2 e^{-3x}$$

$$y'(0) = -2c_1 e^0 - 3c_2 e^0$$

$$2c_1 - 3c_2 = 1 \quad \text{——— (3)}$$

(2) $\times$ (2) $\Rightarrow$ $2 = 2c_1 + 2c_2$ ——— (4)

(4) + (3) $\Rightarrow$ $2c_1 + 2c_2 - 2c_1 - 3c_2 = 2+1$

$$-c_2 = 3$$

$$\boxed{c_2 = -3}$$

(2) $\Rightarrow$ $c_1 - 3 = 1$

$$\boxed{c_1 = 4}$$

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$y = 4e^{-2x} - 3e^{-3x}$ is the Particular solution of (1).

(d). Given $y'' + y' = 0$ ——— (1)

$y_1 = 1$ | $y_2 = e^{-x}$

$y_1' = 0$ | $y_2' = -e^{-x}$

$y_1'' = 0$ | $y_2'' = e^{-x}$

(1) $\Rightarrow y'' + y_1' = 0 \Rightarrow 0 + 0 = 0$

Hence $y_1 = 1$ is the solution of (1)

(1) $\Rightarrow y_2'' + y_2' = 0 \Rightarrow e^{-x} - e^{-x} = 0$

Hence $y_2 = e^{-x}$ is the solution of (1)

$$w(y_1, y_2) = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x} + 0$$

$$= -e^{-x} \neq 0.$$

$w(y_1, y_2) \neq 0$

Hence $y_1$ and $y_2$ are linearly independent

Comparing (1) with Second order differential equation

$$y'' + P(x) y' + Q(x) y = 0$$

$P(x) = 1$ and $Q(x) = 0$.

$P(x)$ and $Q(x)$ are Continuous function.

By theorem,

$y = c_1 (1) + c_2 e^{-x}$ is a general solution for (1)

$$y(2) = c_1 + c_2 e^{-2}$$

$$0 = c_1 + c_2 e^{-2} \quad \text{—— (2)}$$

$$y'(x) = 0 - c_2 e^{-x}$$

$$y'(2) = -c_2 e^{-2} \quad \text{—— (3)}$$

$$\cancel{e^{-2}} = -c_2 \cancel{e^{-2}}$$

$$-c_2 = 1$$

$$\therefore \boxed{c_2 = -1}$$

(2) $\Rightarrow$ $\boxed{c_1 = e^{-2}}$

$y = e^{-2} + (-1) e^{-x}$ is the Particular solution of (1).

Q. Pb. (8) Verify that $y_1 = 1$ ; $y_2 = \log x$ are linearly independent solutions of the equations $y'' + (y')^2 = 0$ on any interval

Is $y = c_1 + c_2 \log x$ the general solution.

Soln:

Given $y'' + (y')^2 = 0$

$x^{-1}$

$-x^{-2}$

| | |
|---|---|
| $y_1 = 1$ | $y_2 = \log x$ |
| $y_1' = 0$ | $y_2' = \frac{1}{x}$ |
| $y_1'' = 0$ | $y_2'' = \frac{-1}{x^2}$ |

(1) $\Rightarrow y_1'' + (y_1')^2 \Rightarrow 0 + 0 = 0$

Hence, $y_1$ is the soln of eqn (1)

(1) $\Rightarrow y_2' + (y_2')^2 = \frac{-1}{x^2} + \left(\frac{1}{x}\right)^2$

$= \frac{-1}{x^2} + \frac{1}{x^2} = 0$

Hence $y_2$ is the solution of equation (1).

$$w(y_1, y_2) = \begin{vmatrix} 1 & \log x \\ 0 & \frac{1}{x} \end{vmatrix}$$

$= \frac{1}{x} - 0 = \frac{1}{x} \neq 0.$

$y_1$ and $y_2$ are linearly independent

$y_1 = c_1 + c_2 \log x$

$y' = 0 + c_2\left(\frac{1}{x}\right)$

$y'' = -c_2 \frac{1}{x^2}$

$(1) \Rightarrow y'' + (y')^2 = \frac{-c_2}{x^2} + \left(\frac{c_2}{x}\right)^2 = 0$

$y = c_1 + c_2 \log x$ is the general solution.

23/7/19 **Theorem:-**

If $y_1$ is a solution of $y'' + P(x)y' + Q(x)y = 0$, then $y_2 = vy_1$ is other independent solution, where $v = \int \frac{1}{y_1^2} e^{-\int P dx} dx$.

**Proof:** $y'' + P(x)y' + Q(x)y = 0$ —— (1)

$y_1$ is a solution of (1)

$y_1'' + P(x)y_1' + Q(x)y_1 = 0$ —— (2)

Assume that $y_2 = vy_1$ is another independent solution of (1)

$y_2' = vy_1' + v'y_1$

$y_2'' = vy_1'' + v'y_1' + v'y_1' + v''y_1$

$y_2'' = vy_1'' + 2v'y_1' + v''y_1$

$y_2$ is a solution of (1)

$$y_2'' + p(x)\, y_2' + Q(x)\, y_2 = 0$$

$$\Rightarrow v y_1'' + 2v' y_1' + v'' y_1 + p(x)(v y_1' + v' y_1)$$
$$+ Q(x)\, v y_1 = 0$$

$$\Rightarrow v'' y_1 + v'(2y_1' + p(x)\, y_1) +$$
$$v(y_1'' + p(x)\, y_1' + Q(x)\, y_1) = 0$$

$$\Rightarrow v'' y_1 + (2y_1' + p(x)\, y_1)\, v' + 0 = 0$$

$$\Rightarrow v'' y_1 + (2y_1' + p(x)\, y_1)\, v' = 0$$

$$\Rightarrow v'' y_1 = -(2y_1' + p(x)\, y_1)\, v'$$

$$\Rightarrow \frac{v''}{v'} = \frac{-2y_1'}{y_1} - \frac{p(x)\, y_1}{y_1}$$

$$= \frac{-2y_1'}{y_1} - p(x)$$

$$\Rightarrow \int \frac{v''}{v'}\, dx = -2 \int \frac{y_1'}{y_1}\, dx + \int p(x)\, dx$$

$$\log v' = -2 \log y_1 - \int p(x)\, dx$$

$$\log v' + 2 \log y_1 = -\int p(x)\, dx$$

$$\log v' + \log y_1^2 = -\int p(x)\, dx$$

$$\log (v' y_1^2) = -\int p(x)\, dx$$

$$v' y_1^2 = e^{-\int p(x)\, dx}$$

$$v' = \frac{1}{y_1^2} e^{-\int p(x)\, dx}$$

$$\boxed{v = \int \frac{1}{y_1^2} e^{-\int p(x)\, dx} \cdot dx} \quad 4.$$

---

$y_1 = x$ is a solution of $x^2 y'' + x y' - y = 0$. Find the general solution.

<u>Soln:</u> Given,

$$x^2 y'' + x y' - y = 0 \quad \text{——— (1)}$$

$y_1 = x$ is a solution of (1)

$\therefore$ Another independent solution is given by

$$y_2 = v y_1 \quad \text{where } v = \int \frac{1}{y_1^2} e^{-\int p(x)\, dx} \cdot dx$$

Now (1) $\Rightarrow$ $$y'' + \frac{1}{x} y' - \frac{y}{x^2} = 0$$

$$\therefore p(x) = \frac{1}{x}$$

$$\therefore v = \int \frac{1}{x^2} e^{-\int \frac{1}{x} dx} \cdot dx$$

$$= \int \frac{1}{x^2} e^{-\log x} \cdot dx$$

$$= \int \frac{1}{x^2} e^{\log x^{-1}} dx$$

$$= \int \frac{1}{x} \cdot x^{-1} dx$$

$$= \int \frac{dx}{x^3}$$

$$= \int x^{-3} dx$$

$$= \left[ \frac{x^{-2}}{-2} \right]$$

$$= -\frac{1}{2x^2}$$

$$\therefore y_2 = vy_1 = \left( -\frac{1}{2x^2} \right) x$$

$$= \left( -\frac{1}{2x} \right)$$

$\therefore$ The general solution of (1)

is $\quad y = c_1 y_1 + c_2 y_2$

$$y = c_1 x - c_2 \frac{1}{2x}$$

Find the general solution of $y''+y=0$. Given that $y_1 = \sin x$ is a solution.

Soln.

Let $y''+y=0$ —— ①

Given $y_1 = \sin x$ is a solution of equation ①. —— ②

Let $y_2 = v y_1$ is another independent solution of equation ①

where $v = \int \frac{1}{y_1^2} e^{-\int p dx} . dx$

From ①,

$P(x) = 0$ and $Q(x) = 1$

$$\therefore v = \int \frac{1}{(\sin x)^2} e^{-\int (0) dx} . dx$$

$$= \int \frac{1}{\sin^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx \quad \left[\because \int \operatorname{cosec}^2 x = -\cot x\right]$$

$$v = -\cot x$$

② $\Rightarrow$ $y_2 = (-\cot x) \sin x$

$$= -\frac{\cos x}{\sin x} \cdot \sin x$$

Solution of ①, $y_2 = -\cos x$ is another indepent

Then $y = c_1 \sin x - c_2 \cos x$ is a general solution of equation ①.

---

P6: The equation $xy'' + 3y' = 0$ has obvious solution $y_1 = 1$, find $y_2$ and general solution.

Soln:

Let $xy'' + 3y' = 0$ ——— ①

Given $y_1 = 1$ is a obvious solution of equation ①.

Let $y_2 = Vy_1$ be another solution of equation ①.

where $V = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx} \cdot dx$

Now comparing the equation ① with general second order differential equation.

$y'' + \frac{3}{x} y' = 0$

Now, we get $p(x) = \frac{3}{x}$ and

$Q(x) = 0$.

Now, comparing the equation

① with general second order differential equation

$$y'' + \frac{3}{x} y' = 0$$

Now, we get $P(x) = \frac{3}{x}$ and

$$Q(x) = 0.$$

Now, we get P

$$\text{Now, } v = \int \frac{1}{(1)^2} e^{-\int 3/x \, dx} \, dx$$

$$= \int e^{-3\int \frac{1}{x} dx} \, dx$$

$$= \int e^{-3\log x} \, dx$$

$$= \int e^{\log x^{-3}} \, dx$$

$$= \int x^{-3} \, dx$$

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$$= \left( \frac{x^{-3+1}}{-3+1} \right) = \frac{x^{-2}}{-2}$$

$$= 1/2 \, x^{-2}$$

$$v = -1/2x^2$$

$$\therefore y_2 = \frac{-1}{2x^2}(y_1) = \frac{-1}{2x^2}(1)$$

$y_2 = \frac{-1}{2x^2}$ is a solution of equation (1).

The general solution of eqn (1)

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 (1) + c_2 \left( \frac{-1}{2x^2} \right)$$

$$y = c_1 - \frac{c_2}{2x^2}$$

---

Pb. Verify that $y_1 = x^2$ is a solution of $x^2y'' + xy' - 4y = 0$. Find $y_2$ and general solution.

Soln.

Let $x^2y'' + xy' - 4y = 0$

Given $y_1 = x^2$

$y_1' = 2x$ and $y_1'' = 2$

Now, $x^2 y_1'' + xy_1' - 4y_1$

$= x^2(2) + x(2x) - 4x^2$

$= 2x^2 + 2x^2 - 4x^2$

$= 0$

Hence $y_1 = x^2$ is a solution of eqn ①.

Let $y_2 = vy_1$ be another soln of eqn ①, where

$$v = \int \frac{1}{y_1^2} e^{-\int p(x)} \cdot dx.$$

Now comparing the eqn ① with second order differential eqn

$$y'' + \frac{x}{x^2} y' - \frac{4}{x} y = 0$$

$p(x) = \frac{1}{x}$ and $Q(x) = \frac{-4}{x}$

Now $v = \int \frac{1}{(x^2)^2} e^{-\int 4/x \, dx} \cdot dx$

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$$= \int \frac{1}{x^4} e^{-\log x} dx$$

$$= \int \frac{1}{x^4} x^{-1} dx$$

$$= \int x^{-4} \cdot x^{-1} dx$$

$$= \int x^{-5} dx$$

$$= \frac{x^{-5+1}}{-5+1} = \frac{x^{-4}}{-4}$$

$$\boxed{V = \frac{-1}{4x^4}}$$

$$\therefore y_2 = \frac{-1}{4x^4} \cdot x^2$$

$y_2 = -\frac{1}{4x^2}$ is a soln of eqn

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 x^2 - \frac{c_2}{4x^2}$$

---

pb: Show that $y_1 = x$ is a soln of the equation $x^2 y'' + 2xy' - 2y = 0$. Find $y_2$ and General soln

Soln: Let $x^2 y'' + 2xy' - 2y = 0$ ——— (1)

Given $y_1 = x$ is the soln of

the eqn ①

Let $y_2 = v y_1$ be another soln

of equation ① where

$$v = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx} \, dx$$

Now comparing eqn ① to second

order differential equation.

$$y'' + \frac{2x}{x^2} y' - \frac{2}{x^2} y = 0$$

$$p(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = \frac{-2}{x^2}$$

$$v = \int \frac{1}{x^2} e^{-\int 2/x \, dx} \, dx$$

$$= \int \frac{1}{x^2} e^{-2\log x} \, dx$$

$$= \int \frac{1}{x^2} e^{\log x^{-2}} \, dx$$

$$= \int \frac{1}{x^2} x^{-2} \, dx$$

$$= \int \frac{1}{x^4} \, dx = \int x^{-4} \, dx$$

$$= \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3}$$

$$v = \frac{-x^{-3}}{3}$$

$$v = \frac{-1}{3x^3} . x,$$

$y_2 = \frac{-1}{3x^2}$ is solution of eqn

$$y = c_1 y_1(x) + c_2 y_2(x)$$

$y = c_1 x - \frac{c_2}{3x^2}$ is a general solution //.

---

Pb: Show that $y_1 = x$ is a solution equation $x^2 y'' - x(x+2)y' + (x+2)y = 0$. Find the general ~~sol~~ equation.

Soln:

Let $x^2 y'' - x(x+2)y' + (x+2)y = 0$ —— (1)

Given $y_1 = x$ is a soln of (1)

Let $y_2 = v y_1$ is another soln of (1)

where $v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx} \, dx$

Now comparing equation (1) with Second order differential equation

$$P(x) = \frac{-(x)(x+2)}{x^2} \quad \text{and}$$

$$Q(x) = \frac{x+2}{x^2}$$

$$P(x) = \frac{-(x+2)}{x} = -1 - \frac{2}{x}$$

$$v = \int \frac{1}{x^2} e^{-\int -(1+2/x) \cdot dx} \cdot dx$$

$$= \int \frac{1}{x^2} \cdot e^{\int (1+2/x)\, dx} \cdot dx$$

$$= \int \frac{1}{x^2} e^{\int dx + 2\, dx/x} \cdot dx$$

$$= \int \frac{1}{x^2} \cdot e^{\int dx + 2 \log x} \cdot dx$$

$$= \int \frac{1}{x^2} e^{x + 2\log x} \cdot dx$$

$$= \int \frac{1}{x^2} e^{x} \cdot e^{\log x^2} \cdot dx$$

$$= \int \frac{1}{x^2} e^{x} \cdot x^2 \cdot dx$$

$$= \int e^{x} \cdot dx = e^{x}$$

$$\boxed{v = e^{x}}$$

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$y_2 = e^x \cdot x$ is a soln of (1)

$y = c_1 y_1(x) + c_2 y_2(x)$

$y = c_1 x + c_2 (e^x \cdot x)$ is a general solution of (1).

---

P6: Verify $y_1 = x^{-1/2} \cdot \sin x$ is one solution of $x^2 y'' + xy' + (x^2 - 1/4) y = 0$. Find the general solution.

Soln.

Let $x^2 y'' + xy' + (x^2 - 1/4) y = 0$.

Given $y = x^{-1/2} \cdot \sin x$

$y_1' = x^{-1/2} \cdot \cos x + \sin x \left(-1/2 \, x^{-1/2 - 1}\right)$

$y_1' = -1/2 \, x^{-3/2} \cdot \sin x + x^{-1/2} \cos x$

$y_1'' = -\frac{1}{2} \left[ -3/2 \, x^{-3/2 - 1} \sin x + x^{-3/2} (\cos x) \right]$

$\left[ -1/2 \, x^{-1/2 - 1} \cos x + x^{-1/2} (-\sin x) \right]$

$= 3/4 \, x^{-5/2} \sin x - 1/2 \, x^{-3/2} \cos x$

$- 1/2 \, x^{-3/2} \cos x - x^{-1/2} \sin x.$

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$$y_1'' = 3/4\, x^{-5/2} \sin x - x^{-3/2} \cos x - x^{-1/2} \sin x.$$

$$(1) \Rightarrow x^2 y_1'' + x y_1' + (x^2 - 1/4) y_1$$

$$= x^2 \left[ 3/4\, x^{-5/2} \sin x - x^{-3/2} \cos x - x^{-1/2} \sin x \right]$$

$$+ x \left[ -1/2\, x^{-3/2} \sin x + x^{-1/2} \cos x \right]$$

$$+ (x^2 - 1/4)(x^{-1/2} \sin x)$$

$$= 3/4\, x^2 . x^{-5/2} \sin x - x^2 . x^{-3/2} \cos x - x^2 . x^{-1/2} \sin x + x . x^{-1/2} . \sin x + 1/4\, x^{-1/2} \sin x.$$

$$= 3/4\, x^{-1/2} \sin x - x^{1/2} \cos x - x^{3/2} \sin x + x^{1/2} \cos x - 1/2 \sin x . x^{-1/2} + x^{3/2} . \sin x - 1/4 . x^{-1/2} \sin x$$

$$= 3/4\, x^{-1/2} \sin x - 1/2\, x^{-1/2} \sin x - 1/4\, x^{-1/2} \sin x .$$

$$= -1/2\, x^{-1/2} . \sin x + 1/2\, x^{-1/2} \sin x$$

$y_1 = 0.$

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Hence $y_1 = x^{-1/2} \cdot \sin x$ is the solution of eqn (1)

let $y_2 = vy_1$ be another soln of (1),

where $v = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx} \cdot dx$

Now Comparing equation (1) to Second order differential equation.

$$y_1'' + \frac{x}{x^2} y_1' + \frac{(x^2 - 1/4)}{x^2} y_1 = 0.$$

$p(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2 - 1/4}{x^2}$

$$v = \int \frac{1}{(x^{-1/2} \sin x)^2} e^{-\int 1/x \cdot dx} \cdot dx$$

$$= \int \frac{1}{x^{-1} \sin^2 x} e^{-\log x} \cdot dx$$

$$= \int \frac{x}{\sin^2 x} e^{\log x^{-1}} \cdot dx$$

$$= \int \frac{x+1}{\sin^2 x} x^{-1}\,dx$$

$$= \int \frac{1}{\sin^2 x}\,dx$$

$$= \int \operatorname{cosec}^2 x \, dx$$

$$\boxed{v = -\cot x}$$

$$y_2 = v y_1 = -\cot x \cdot x^{-1/2} \sin x$$

$$= \frac{-\cos x}{\sin x} x^{-1/2} \cdot \sin x$$

$y_2 = -x^{-1/2} \cos x$ is a soln of (1)

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 \left(x^{-1/2} \sin x\right) - c_2 \left(x^{-1/2} \cos x\right)$$

---

Pb: Verify $y_1 = e^x$ is a soln of $xy'' - (2x+1)y' + (x+1)y = 0$. Find $y_2$ and general solution.

Soln:

Let $xy'' - (2x+1)y' + (x+1)y = 0$ —— (1)

Given $y_1 = e^x$

$y_1' = e^x$ and $y_2'' = e^x$

(1) $\Rightarrow xy_1'' - (2x+1)y_1' + (x+1)y_1$

$= x\,(e^x) - (2x+1)\,e^x + (x+1)\,e^x$

$= x e^x - 2x e^x - e^x + x e^x + e^x$

$= 0$

$\therefore$ Hence $y_1 = e^x$ is the soln of (1)

let $\underline{y_2 = v y_1}$ is $\underline{\text{another}}$ solution of (1)

where $v = \int \frac{1}{y_1^2}\, e^{-\int p(x)\,dx} \cdot dx$

Comparing eqn (1) with second order differential equation.

$$y'' - \frac{(2x+1)}{x} y' + \frac{(x+1)}{x} y = 0$$

$$p(x) = -(2 + 1/x) \quad \text{and} \quad Q(x) = \frac{x+1}{x}$$

$$v = \int \frac{1}{(e^x)^2} \cdot e^{-\int -(2+1/x)\,dx} \cdot dx$$

$$= \int \frac{1}{e^{2x}} \cdot e^{\int (2+1/x)\,dx} \cdot dx$$

$$= \int \frac{1}{e^{2x}} \cdot e^{2x} \cdot e^{\log x} \cdot dx$$

$$= \int x\,dx$$

$$\boxed{v = \frac{x^2}{2}}$$

$y_2 = \frac{x^2}{2}$ is the soln of (1)

The general soln is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 (e^x) + c_2 \left(\frac{x^2 e^x}{2}\right)$$

---

Pb: If $y_1$ is a non-zero soln of eqn. $y'' + p(x) y' + Q(x) y = 0$ and $y_2 = v y_1$. where $v = \int \frac{1}{y_1^2} e^{-\int p(x) dx} \cdot dx$

Show that wronskian of $y_1$ and $y_2$ are linearly independent

Soln:

let $y'' + p(x) y' + Q(x) y = 0$

and $v = \int \frac{1}{y_1^2} e^{\int p(x) dx} \cdot dx$

$w(y_1, y_2)$ is linearly independent

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Since $y_2 = v y_1$

$$y_2' = v y_1' + v' y_1$$

$$= y_1 (v y_1' + v' y_1) - y_1' (v y_1)$$

$$= y_1 v y_1' + y_1^2 \cdot v' - y_1 v'$$

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$= y_1 \cdot v y_1' + y_1^2 v' - y_1 v'$

$w(y_1, y_2) = v' y_1^2 \neq 0.$

$y_1$ and $y_2$ are linearly independent

---

Pb: The equation $(1-x^2)y'' - 2xy' + 2y = 0$ is the special case of Legendre's eqn. $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ corresponding to $p=1$. It has $y_1 = x$ as a obvious soln. Find the general soln.

<u>Soln:</u>

Let $(1-x^2)y'' - 2xy' + 2y = 0$ —— ①

Given $y = x$ is an obvious soln of ①.

Let $y_2 = vy_1$ is another ~~soln~~ of independent soln ~~of~~

where $v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx} \, dx$

Comparing eqn ① is second order differential equation

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{2y}{(1-x^2)} = 0$$

$$P(x) = \frac{-2x}{1-x^2} \qquad Q(x) = \frac{2y}{1-x^2}$$

Now, $v = \int \frac{1}{x^2} e^{-\int \frac{-2x}{1-x^2} dx} \, dx$

$$= \int \frac{1}{x^2} e^{-\log_e (1-x^2)} \, dx$$

$$= \int \frac{1}{x^2} e^{\log (1-x^2)^{-1}} \, dx$$

$$= \int \frac{1}{x^2} (1-x^2)^{-1} \, dx$$

$$= \int \frac{1}{x^2} \frac{1}{(1-x^2)} \, dx$$

$$= \int \frac{1}{x^2 (1-x^2)} \, dx \quad \text{--- (2)}$$

$I = \int f(x) \sin x \, dx$

Consider,

$$\frac{1}{x^2(1-x^2)} = \frac{A}{x^2} + \frac{B}{1-x^2}$$

$$\frac{1}{x^2(1-x^2)} = \frac{A(1-x^2) + B x^2}{x^2(1-x^2)}$$

put $x = 1$ $\quad 1 = A(0) = B$

29/7/19

# Method of Variation of Parameters

To find Particular soln of second order differential equation

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{---} (1)$$

To find the function $v_1$ and $v_2$

$$y = v_1 y_1 + v_2 y_2 \quad \text{---} (2)$$

Now, consider,

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{and} \quad (3)$$

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{---} (4)$$

$$y' = v_1 y_1' + \underline{v_1' y_1} + v_2 y_2' + \underline{v_2' y_2} \quad \text{---} (5)$$

Let $\quad v_1' y_1 + v_2' y_2 = 0 \quad \text{---} (6)$

then $\quad y' = v_1 y_1' + v_2 y_2' \quad \text{---} (7)$

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' \quad \text{---} (8)$$

$(1) \Rightarrow v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'$
$\quad + P(x)(v_1 y_1' + v_2 y_2') + Q(x)(v_1 y_1 + v_2 y_2) = R(x)$

$v_1 ( y_1'' + p(x) y_1' + Q(x) y_1 )$

$+ v_2 ( y_2'' + p(x) y_2' + Q(x) y_2 )$

$+ v_1' y_1' + v_2' y_1' = R(x)$

$v_1 (0) + v_2 (0) + v_1' y_1' + v_2' y_2' = R(x)$

$\{ \because y_1$ and $y_2$ are solution of (1) $\}$

$\Rightarrow \quad v_1' y_1' + v_2' y_2' = R(x)$ —— (9)

From (6) and (9)

$$\begin{array}{cccc} & v_1' & v_2' & 1 \\ y_2 & 0 & y_1 & y_2 \\ y_2' & -R(x) & y_1' & y_2' \end{array}$$

$$\frac{v_1'}{-R(x) y_2} = \frac{v_2'}{R(x) y_1} = \frac{1}{y_1 y_2' - y_2 y_1'}$$

$$\therefore v_1' = \frac{-R(x) y_2}{y_1 y_2' - y_2 y_1'}$$

$$\int v_1' \, dx = \int \frac{-R(x) y_2}{y_1 y_2' - y_1' y_2} \, dx$$

$$\boxed{v_1 = -\int \frac{R(x) y_2}{y_1 y_2' - y_1' y_2} . dx}$$

$$v_2' = \frac{R(x) y_1}{y_1 y_2' - y_1' y_2}$$

$$\int v_2' \, dx = \int \frac{R(x) y_1}{y_1 y_2' - y_1' y_2} . dx$$

$W = y_1 y_2' - y_1' y_2$

$$\boxed{v_2 = \int \frac{R(x) y_1}{y_1 y_2' - y_1' y_2} . dx} = \int \frac{R(x) y_1}{W}$$

(1) Find Particular soln of

$y'' + y = \operatorname{cosec} x$ .

Soln: $y'' + y = \operatorname{cosec} x$ —— (1)

Now corresponding homogeneous equation is

$y'' + y = 0$

A.E is $m^2 + 1 = 0 \Rightarrow m^2 = -1$

$m = \pm i$

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

C.F is

$y = c_1 \cos x + c_2 \sin x$

Now, $y_1 = \cos x$; $y_2 = \sin x$.

$y_1' = -\sin x$, $y_2' = \cos x$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

Hence $R(x) = \operatorname{cosec} x$

$$v_1 = -\int \frac{R(x)\, y_2}{y_1 y_2' - y_1' y_2}\, dx$$

$$= -\int \frac{\operatorname{cosec} x\,(\sin x)}{1}\, dx$$

$$= -\int dx$$

$\left(\because \operatorname{cosec} x = \frac{1}{\sin x}\right)$

$$\boxed{v_1 = -x}$$

$|a_r - a_m| \leq \ell$

$|a_r - \ell| < k$

$$v_2 = \int \frac{R(x)\, y_1}{y_1 y_2' - y_2 y_1'}\, dx$$

$$= \int \frac{\operatorname{cosec} x\, (\cos x)}{1}\, dx$$

$$= \int \frac{\cos x}{\sin x}\, dx$$

$$= \int \cot x\, dx$$

$$\boxed{v_2 = \log \sin x}$$

Hly $\int \frac{\sin}{\cos x}$ $\Downarrow$ $\log \cos x$

$\therefore$ The Particular solution of (1)

is $y = v_1 y_1 + v_2 y_2$

$y = -x \cos x + (\log \sin x) \sin x$

---

pb:- Find Particular soln of

$y'' - 2y' + y = 2x$.

Soln: Given that $y'' - 2y' + y = 2x$ — (1)

corresponding to the homogenous eqn of (1)

$$y'' - 2y' + y = 0$$

Auxilary eqn (1) is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

C.F is $y = e^{x}(c_1 x + c_2)$

$$y = c_1 x e^{x} + c_2 e^{x}$$

$y_1 = xe^{x}$ , $y_2 = e^{x}$

$$w(y_1, y_2) = \begin{vmatrix} xe^{x} & e^{x} \\ xe^{x}+e^{x} & e^{x} \end{vmatrix}$$

$$= xe^{2x} - xe^{2x} - e^{2x}$$

$$= -e^{2x}$$

i.e) $y_1 y_2' - y_1' y_2 = -e^{2x}$

$$v_1 = \int \frac{-R(x) y_2}{y_1 y_2' - y_1' y_2} dx$$

$$= -\int \frac{2x \cancel{x} e^{x}}{-e^{2x}} dx$$

$$= 2\int \frac{xe^{x}}{e^{2x}} dx$$

$$= 2\int \frac{x}{e^{x}} dx$$

$$= 2\int xe^{-x} dx$$

$\int u dv = uv - \int v du$

$$v_1 = 2\left[-xe^{-x} - e^{-x}\right]$$

$$v_2 = \int \frac{R(x)\, y_1}{y_1 y_2' - y_1' y_2}\, dx$$

$$= \int \frac{2x \cdot xe^{x}}{-e^{2x}}\, dx$$

$$= -2 \int x^2 e^{-x}\, dx$$

$$= -2\left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]$$

$$v_2 = 2x^2 e^{-x} + 4xe^{-x} + 4e^{-x} + c$$

The Particular soln of (1)

$$y = v_1 y_1 + v_2 y_2$$

$$= \left[2\left(-xe^{-x} - e^{x}\right)\right] xe^{x}$$

$$+ \left[2x^2 e^{-x} + 4xe^{-x} + 4e^{-x}\right] e^{x}$$

$$= -2xe^{-x} \cdot xe^{x} - 2x \cdot xe^{x}$$

$$+ 2x^2 + 4x + 4$$

$$= -2x^2 - 2x + 2x^2 + 4x + 4$$

$$y = 2x + 4 = 2(x+2)$$

P6. Find the particular soln using method of variation of Parameter.

Soln: (a) $y'' - y' - 6y = e^{-x}$

(b) $y'' + 4y = \tan 2x$

(c) $y'' + 2y' + 5y = e^{-x} \sec 2x$

(d) $y'' + y = \sec x$

(e) $y'' + y = x \cos x$

Soln.

(a) Given $y'' - y' - 6y = e^{-x}$ —— (1)

Corresponding homgeneous eqn is $y'' - y' - 6y = 0$ —— (2)

To find y

$$m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3 \quad m = -2$$

$$y = c_1 e^{3x} + c_2 e^{-2x}$$

$$W(y_1, y_2) = -e^{3x} \cdot 2e^{-2x} - 3e^{-2x} \cdot e^{3x}$$

$$= -2e^{x} - 3e^{x} = -5e^{x}$$

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$$v_1 = \int \frac{-R(x)\, y_2}{y_1 y_2' - y_2 y_1'}\, dx$$

$$= -\int \frac{e^{-x} \cdot e^{-2x}}{-5e^{x}}\, dx$$

$$= \frac{1}{5} \int \frac{e^{-3x}}{e^{x}}\, dx$$

$$v_1 = \frac{1}{5} \left[ \frac{e^{-4x}}{-4} \right]$$

$$v_1 = \frac{-1}{20} e^{-4x}$$

$$v_2 = \int \frac{R(x)\, y_1}{w(y_1, y_2)}\, dx$$

$$= \int \frac{e^{-x} e^{2x}}{-5e^{x}}\, dx$$

$$= -1/5 \int \frac{e^{x}}{e^{x}}\, dx$$

$$v_2 = -1/5\, x$$

$$y = v_1 y_1 + v_2 y_2$$

$$= -1/2e\, e^{-4x} \cdot e^{3x} + 1/5\, e^{-2x} \cdot e^{x}$$

$$= \frac{-e^{-x} + 4e^{-x}}{20}$$

$$y = \frac{-5e^{-x}}{20}$$

$$y = \frac{-1}{4} e^{-x}$$

[b] $y'' + 4y = \tan 2x$ —— (1)

corresponding homogenous eqn is

$$y'' + 4y = 0 \quad \text{—— (2)}$$

$$m^2 + 4 = 0$$

$$m^2 = -2$$

$$m = \sqrt{-2}$$

$$m = \pm 2i$$

C.F $y = e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$

$$= c_1 \cos 2x + c_2 \sin 2x$$

$y_1 = \cos 2x$ $\qquad$ $y_2 = \sin 2x$

$y_1' = 2 \sin 2x$ $\qquad$ $y_2' = 2 \cos 2x$

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$$w(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ 2\sin x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos 2x \cdot \cos 2x - \sin 2x \cdot 2\sin x$$

$$= 2(1) = 2$$

now, $$v_1 = \int \frac{-R(x)\, y_2}{y_1 y_2' - y_2 y_1'}\, dx$$

$$= \int \frac{-\tan 2x \cdot \sin 2x}{2}\, dx$$

$$= -\frac{1}{2} \int \tan 2x \cdot \sin 2x$$

$$= -\frac{1}{2} \int \frac{\sin 2x}{\cos 2x} \cdot \sin 2x\, dx$$

$$= -\frac{1}{2} \int \frac{\sin^2 2x}{\cos 2x}\, dx$$

$$= -\frac{1}{2} \int \frac{(1 - \cos^2 2x)}{\cos 2x}\, dx$$

$$= -\frac{1}{2} \int \left( \frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x} \right) dx$$

$$= -\frac{1}{2}\int \frac{1}{\cos 2x} - \cos 2x \, dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x)\, dx$$

$$= \frac{-1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$v_1 = \frac{-1}{4}\left[(\log \sec 2x + \tan 2x) - \sin 2x\right]$$

$$v_2 = \int \frac{R(x) y_1}{y_1 y_2' - y_2 y_1'} \cdot dx$$

$$= \int \frac{\tan 2x \cdot \cos 2x}{2}\, dx.$$

$$= \frac{1}{2}\int \frac{\sin 2x}{\cos 2x} \cdot \cos 2x \cdot dx$$

$$= +\frac{1}{2}\int \frac{\sin 2x}{\cos 2x} \cdot \cancel{\cos 2x} \cdot dx$$

$$= \frac{+1}{2}\int \sin 2x \cdot dx$$

$$= \frac{-1}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$v_2 = \frac{-1}{4}\cos 2x$$

Particular soln is

$$y = v_1 y_1 + v_2 y_2$$

$$= \frac{-1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] \cos 2x$$

$$+ \left(\frac{-1}{4}\cos 2x\right)\sin 2x$$

$$= \frac{-1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

$$+ \frac{1}{4}\cos 2x \cdot \sin 2x - \frac{1}{4}\cos 2x \cdot \sin 2x$$

$$y = \frac{-1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

(C) $y'' + 2y' + 5y = e^{-x} \cdot \sec 2x$ ——— (1)

Soln.

Given $y'' + 2y' + 5y = e^{-x} \cdot \sec 2x$

The corresponding given homogenous equation is $y'' + 2y' + 5y = 0$.

The Auxillary eqn is

$$m^2 + 2m + 5 = 0$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$m = \frac{-2 + 4i}{2} = -1 \pm 2i$$

The roots are imaginary C.F is

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y = c_1 e^{-x} . \cos 2x + c_2 e^{-x} \sin 2x$$

$$y_1 = e^{-x} . \cos 2x \ ; \quad y_2 = e^{-x} . \sin 2x$$

$$y_1' = e^{-x} . (-\sin 2x) . 2 + \cos 2x \, (-e^{-x})$$

$$y_1' = -2e^{-x} \sin 2x - e^{-x} \cos 2x$$

$$y_2' = e^{-x}(2 \cos 2x) + \sin 2x \, (-e^{-x})$$

$$= 2e^{-x} . \cos 2x - e^{-x} \sin 2x$$

$$w(y_1, y_2) = \begin{vmatrix} e^{-x} \cos 2x & e^{-x} \sin 2x \\ -2e^{-x} \sin 2x - e^{-x} \cos 2x & 2 e^{-x} \cos 2x - e^{-x} \sin 2x \end{vmatrix}$$

$$= e^{-x} . \cos 2x \left[ 2 e^{-x} . \cos 2x - e^{-x} \sin 2x \right]$$

$$- e^{-x} \sin 2x \left[ -2e^{-x} \sin 2x - e^{-x} \cos 2x \right]$$

$$= (e^{-x} . \cos 2x . 2e^{-x} . \cos 2x - e^{-x} e^{-x} \sin 2x$$

$$(e^{-x} \cos 2x . 2e^{-x} \cos 2x \cos 2x - e^{-x} e^{-x} \sin 2x . \cos 2x)$$

$$= 2e^{-2x}\cos^2 2x - \cancel{e}^{-2x}\sin 2x +$$

$$2e^{-x}\sin^2 2x + e^{-2x}\cos 2x.$$

$$= -e^{-x}\cdot\cos 2x\left[e^{-x}(\sin 2x - 2\cos 2x)\right]$$

$$+ e^{-x}\sin 2x\left[\cancel{2}e^{-\cancel{2}x}(\cos 2x + 2\sin 2x)\right]$$

$$= -e^{-2x}\cdot\cos 2x\cdot\sin 2x + 2e^{-2x}\cdot\cos^2 2x$$

$$+ e^{-2x}\sin 2x\cos x + 2e^{-2x}\cdot\sin^2 2x$$

$$= 2e^{-2x}(\cos^2 x + \sin^2 2x)$$

$$W(y_1, y_2) = -2e^{-2}\cdot x$$

$$v_1 = \int \frac{-R(x)y_2}{y_1 y_2' - y_2 y_1'}\cdot dx$$

$$= \int \frac{e^{-x}\cdot\sec 2x\cdot e^{-x}\cdot\sin 2x}{2e^{-x}}\cdot dx$$

$$= \frac{-1}{2}\int \sec 2x - \sin 2x\cdot dx$$

$$= \frac{-1}{2}\int \frac{1}{\cos 2x}\cdot\sin 2x\, dx$$

$$= \frac{-1}{2}\int \tan^2 x\, dx$$

$$v_1 = \frac{-1}{2}\left(-\log \frac{\cos 2x}{2}\right)$$

$$v_2 = \int \frac{-R(x)y_2}{W(y_1, y_2)}\,dx$$

$$= \int \frac{e^{-x}.\sec 2x}{2e^{-2x}}\, e^{-x}.\cos 2x.dx$$

$$= \frac{1}{2}\int \frac{1}{\cos 2x} \times \cos 2x.dx$$

$$= \frac{1}{2}\int dx = \frac{x}{2}$$

$$\boxed{v_2 = x/2}$$

The Particular Soln of (1)

$$y = v_1 y_1 + v_2 y_2$$

$$y = \frac{-1}{4}(-\log \cos 2x)e^{-x}.\cos 2x + \frac{x}{2}e^{-x}\sin 2x$$

$$y = e^{-x}\left(\frac{1}{4}\cos 2x \log 2x + \frac{x}{2}\sin 2x\right)$$

(d) Given $y'' + y = \sec x$

Corresponding homogenous eqn is

$$y'' + y = 0$$

A.E is $m^2+1=0$

$m = \pm i$

C.F is $y = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x$ | $y_2 = \sin x$

$y_1' = -\sin x$ | $y_2' = \cos x$

$w(y_1, y_2) = \cos^2 x + \sin^2 x = 1$

$$v_1 = \int \frac{-R(x) y_2}{w(y_1, y_2)} \cdot dx$$

$$= \int \frac{-\sec x \cdot \sin x}{1} \cdot dx$$

$$= -\int \frac{\sin x}{\cos x} \cdot dx$$

$$= -\int \tan x \, dx$$

$$= -\log \cos x.$$

$$\boxed{v_1 = \log \cos x}$$

$$v_2 = \int \frac{R(x) y_1}{w(y_1, y_2)} \cdot dx$$

$$= \int \sec x \cdot \cos x \, dx$$

$$= \int dx$$

$$\boxed{v_2 = x}$$

$$y = v_1 y_1 + v_2 y_2$$

$y = \log(\cos x) + \cos x + x \sin x$ is the general soln of (1)

(e) $y'' + y = x \cdot \cos x$

Soln Given $y'' + y = \cos x$

corresponding homogenous eqn is

$$y'' + y = 0$$

A.E is $m^2 + 1 = 0$

$$m = \pm i$$

$$y = e^{0x}(c_1 \cos x + c_2 \sin x)$$

$y_1 = \cos x$ $\quad y_2 = \sin x$

$y_1' = -\sin x$ $\quad y_2' = \cos x$

$$w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$v_1 = \int \frac{-R(x)\, y_2}{w(y_1, y_2)} \cdot dx$$

$$= \int \frac{-x \cdot \cos x \cdot \sin x}{1}\, dx$$

$$= \int -x \cos x \sin x \, dx$$

$$= \int -x \frac{\sin 2x}{2}\, dx$$

$$= \frac{-1}{2} \int x \sin 2x \, dx$$

$$= \frac{-1}{2}\left[-x \frac{\cos 2x}{2}\right]$$

$$= -\int \frac{\cos 2x}{2}\, dx$$

$$= -\tfrac{1}{2}\left[\frac{-x \cos 2x}{2}\right] + \frac{1}{2}\,\frac{\sin 2x}{2}$$

$$= \frac{-1}{2}\left[\frac{-x \cos 2x}{2} + \frac{\sin 2x}{2}\right]$$

$$V_1 = \frac{-1}{4}\left[-x \cos 2x + \frac{\sin 2x}{2}\right]$$

$$V_2 = \int \frac{R(x)\, u_1}{W}\, dx = \int \frac{x \cos x \cdot \cos x}{1}\, dx$$

$$= \int x \cos^2 x \, dx = \int \frac{x(1 + \cos 2x)}{2}\, dx$$

$$= \frac{1}{2} \int (x + x \cos 2x)\, dx$$

$$= \frac{1}{2}\int x\,dx + \int x\cos 2x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right] + \int x\cos 2x \cdot dx$$

Consider $\int x\cos 2x \cdot dx$

$$\int u\,dv = uv - \int v\,du$$

$$= x\left(\frac{\sin 2x}{2}\right) + \int \frac{\sin 2x}{2}\,dx$$

$$= \frac{x}{2}\sin 2x + \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x$$

$$\Rightarrow \quad = \frac{1}{2}\frac{x^2}{2} + \frac{1}{2}\left[\frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x\right]$$

$$v_2 = \frac{x^2}{4} + \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x$$

The Particular soln is

$$y = v_1 y_1 + v_2 y_2$$

$$= \frac{-1}{4}\left[-x\cos 2x + \frac{\sin 2x}{2}\right]\cos 2x +$$

$$\left[\frac{x^2}{4} + \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x\right]\cdot \sin x$$

$$y = \cos x \cdot \left( \frac{x}{4} \cos 2x - \frac{\sin 2x}{8} \right)$$

$$+ \left[ \frac{x^2}{4} + \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x \right] \sin x$$

$$= \frac{x}{4} \cos x \cdot \cos 2x - \frac{\cos x \cdot \sin 2x}{8}$$

$$+ \frac{x^2}{4} \sin x + \frac{x}{4} \sin 2x \cdot \sin x$$

$$- \frac{\cos x \cdot \sin x}{8}$$

$$= \frac{x}{4} \left[ \cos x \cdot \cos 2x + \sin x \sin 2x \right]$$

$$- \frac{1}{8} \left[ \sin 2x \cdot \cos x - \sin x \cdot \cos 2x \right]$$

$$+ \frac{x^2}{4} \sin x$$

$$= \frac{1}{4} \left[ x (\cos 2x - x) + x^2 \sin x - \frac{1}{2} \sin 2x - x \right]$$

$$= \frac{x^2}{4} \sin x + \frac{x}{4} \cos x - \frac{\sin x}{8}$$

$$y = \frac{1}{4} \left[ x^2 \sin x + x \cos x - \frac{\sin x}{2} \right]$$

Pbm Find the Particular soln by using the method of Variation of Parameter

$y'' - 2y' - 3y = 64xe^{-x}$

Soln.

Given $y'' - 2y' - 3y = 64xe^{-x}$

corresponding homogenous equation is

$$y'' - 2y' - 3y = 0$$

The A.E is $m^2 - 2m - 3 = 0$

$$m = -1, 3$$

The C.F is

$$y = c_1 e^{-x} + c_2 e^{3x}$$

| | |
|---|---|
| $y_1 = e^{x}$ | $y_2 = e^{3x}$ |
| $y_1' = e^{x}$ | $y_2' = 3e^{3x}$ |
| $y_1'' = e^{x}$ | $y_2'' = 9e^{3x}$ |

$$W(y_1, y_2) = \begin{vmatrix} e^{x} & e^{3x} \\ -e^{x} & 3e^{3x} \end{vmatrix}$$

$$= e^{x}(3e^{3x}) - (-e^{x})(e^{3x})$$

$$= 3e^{2x} + e^{2x} = 4e^{2x}$$

$$v_1 = \int \frac{-R(x)\, y_2}{W(y_1, y_2)} \cdot dx$$

$$= \int \frac{-64\, x e^{-x}}{4 e^{2x}} \cdot e^{3x} \cdot dx$$

$$= \int \frac{-16\, x e^{-x}}{e^{2x}} \cdot e^{3x} \cdot dx$$

$$= -16 \int \frac{x \cdot e^{-x+3x}}{e^{2x}} \cdot dx$$

$$= -16 \int \frac{x \cdot e^{2x}}{e^{2x}}\, dx = -16\left(\frac{x^2}{2}\right) = -8x^2$$

$$\boxed{v_1 = -8x^2}$$

$$v_2 = \int \frac{R(x)\, y_1}{W(y_1, y_2)}\, dx$$

$$= \int \frac{64\, x e^{-x}}{4 e^{2x}} \cdot e^{-x} \cdot dx$$

$$= 16 \int \frac{x \cdot e^{-2x}}{e^{2x}} \cdot dx = 16 \int x e^{-2x} e^{-2x} \cdot dx$$

$$= 16 \int x e^{-4x} \cdot dx$$

$$= 16 \left[ x \cdot \frac{e^{-4x}}{4} \right] - 16 \int \frac{e^{-4x}}{-4}\, dx$$

$$= -\frac{16}{4} \left(x e^{-4x}\right) + 4 \int e^{-4x} \cdot dx$$

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$$= -4\left(xe^{-4x}\right) + 4\left(\frac{e^{-4x}}{4}\right)$$

$$= -4xe^{-4x} + e^{-4x}$$

$$v_2 = -4 xe^{-4x}\left(x + \frac{1}{4}\right)$$

The general soln is

$$y = v_1 y_1 + v_2 y_2$$

$$= (-8x^2)(e^{-x}) + \left[-4e^{-4x}\left(x + \frac{1}{4}\right)\right]e^{3x}$$

$$= -8x^2 \cdot e^{-x} - 4e^{-x}\left(x + 1/x\right)$$

$$= -4e^{-x}\left(\left(x + \frac{1}{4}\right) + 2x^2\right)$$

$$= -4e^{-x}\left[\frac{4x + 1 + 8x^2}{4}\right]$$

$$y = -e^{-x}\left(4x + 8x^2 + 1\right)$$

Pb: Find the general Soln of

$(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2$.

Soln. Given $(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2$ — (1)

corresponding to the homogeneous eqn

$$x(x^2-1)y'' - 2xy' + 2y = 0 \quad \text{--- (2)}$$

$y_1 = x$ is the soln of eqn (2)

$$\boxed{y = x ; y' = 1 \quad \text{and} \quad y'' = 0}$$

Let $y_2 = v y_1$ be another independent soln of 2 where

$$v = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx} \cdot dx$$

$$= \int \frac{1}{x^2} e^{-\int \frac{-2x}{x^2-1} dx} \cdot dx$$

$$= \int \frac{1}{x^2} e^{\int \frac{2x}{x^2-1} dx} \cdot dx$$

$$= \int \frac{1}{x^2} e^{\log(2x^2-1)} dx$$

$$= \int \frac{1}{x^2} (x^2-1)\,dx$$

$$= \int \left(1 - \frac{1}{x^2}\right) dx$$

$$= \int dx - \int \frac{1}{x^2} dx$$

$$= x - \left[\frac{x^{-2+1}}{-2+1}\right]$$

$$= x - \frac{x^{-1}}{}$$

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$$\boxed{v = x + \frac{1}{x}}$$

$y_2 = v y_1 \Rightarrow \left(x + \frac{1}{x}\right) x = x^2+1$

$y_2 = x^2+1$ is the soln of eqn (2)

The general soln of eqn

$$y = c_1 y_1 + c_2 y_2$$

$$y_c = c_1 x + c_2 (x^2+1)$$

let,

| | |
|---|---|
| $y_1 = x$ | $y_2 = x^2+1$ |
| $y_1' = 1$ | $y_2' = 2x$ |

$$w(y_1, y_2) = \begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - (x^2+1)$$

$$= 2x^2 - x^2 - 1 = x^2 - 1$$

$$v_1 = \int \frac{-R(x) y_2}{w(y_1, y_2)} dx$$

$$= \int \frac{-1(x^2-1)}{x^2-1} (x^2+1) dx$$

$$= \int -(x^2+1)\,dx = -\int x^2 dx + \int dx$$

$$v_1 = -\left(\frac{x^3}{3} + x\right)$$

$$v_2 = \int \frac{R(x)\,y}{W(y_1, y_2)}\,dx$$

$$= \int \frac{(x^2-1)}{x^2-1}(x)\,dx = \frac{x^2}{2}$$

$$\boxed{v_2 = \frac{x^2}{2}}$$

The Particular Soln is

$$y = v_1 y_1 + v_2 y_2$$

$$= -\left(\frac{x^3}{3} + x\right)x + \frac{x^2}{2}(x^2+1)$$

$$= -\frac{x^4}{3} - x^2 + \frac{x^4}{2} + \frac{x^2}{2}$$

$$= \frac{-2x^4+3x^4}{6} - \frac{2x^2+x^2}{2}$$

$y = \frac{x^4}{6} - \frac{x^2}{2}$ is the Particular

Soln of ②

$y = c_1 x + c_2 (x^2+1)$ is a general soln of $x^2-1$

$$y'' - 2xy' + 2y = 0$$

Particular soln of $y = v_1 y_1 + v_2 y_2$ is

$$y = \frac{x^4}{6} - \frac{x^2}{2}$$

The general soln of eqn (1)

$$\boxed{y = c_1 x + c_2 (x^2+1) + \frac{x^4}{6} - \frac{x^2}{2}}$$

Pb: Find the general soln of

$(x^2+x) y'' + (2-x^2) y' - (2+x) y = x (x+1)^2$

Soln

Given

$(x^2+x) y'' + (2-x^2) y' - (2+x) y = x(x+1)^2$ — (1)

Corresponding to the homogeneous eqn.

$(x^2+x) y'' + (2-x^2) y' - (2+x) y = 0$

$x^2 e^x + x e^x + 2e^x - x^2 e^x - 2ye^x - xe^x$ — (2)

$y_1 = e^x$ is the soln of eqn (2)

$[y_1 = e^x, \; y_1' = e^x, \; y_1'' = e^x]$

Let $y_2 = v y_1$ be another independent soln of the equation (2),

where $v = \int \frac{1}{y^2} e^{-\int p(x) dx} \cdot dx$

$$= \int \frac{1}{(e^x)^2} e^{-\int \left(\frac{2-x^2}{x^2+x}\right) dx} \cdot dx$$

$$\int \frac{1}{(e^x)^2} e^{\int \frac{x^2-2}{x(x+1)} dx} \cdot dx$$

consider,

$$\frac{x^2-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} + C.$$

$$\frac{x^2-2}{x(x+1)} = \frac{A(x+1) + B(x) + C(x)(x+1)}{x(x+1)}$$

$$x^2-2 = A(x+1) + B(x) + C(x)(x+1)$$

Put $x=0 \Rightarrow 0-2 = A$

$$\boxed{A = -2}$$

Put $x=-1 \Rightarrow -1 = B$

$$\boxed{B = 1}$$

Put $x=1 \Rightarrow 1-2 = A(2) + B(1) + C(1)(2)$

$$2A + B + 2C = -1$$

$$\Rightarrow 2(-2) + 1 + 2C = -1$$

$2c = 2$

$\boxed{c = 1}$

$$= \int \frac{1}{e^{2x}} e^{\int \left(\frac{-2}{x} + \frac{1}{x+1} + 1\right) dx} \cdot dx$$

$$= \int \frac{1}{e^{2x}} e^{\int \frac{-2}{x} dx} \cdot e^{\int \frac{1}{x+1} dx} \cdot e^{\int dx} \cdot dx$$

$$= \int \frac{1}{e^{2x}} \cdot e^{-2\int \frac{dx}{x}} \cdot e^{\log(x+1)} \cdot e^{x} \cdot dx$$

$$= \int \frac{1}{e^{2x}} \cdot e^{-2\log x} \cdot e^{\log(x+1)} \cdot e^{x} \cdot dx$$

$$= \int \frac{1}{e^{2x}} e^{\log x^{-2}} e^{\log(x+1)} \cdot e^{x} \cdot dx$$

$$= \int \frac{1}{e^{2x}} \cdot \frac{1}{x^2} (x+1) \cdot e^{x} \cdot dx$$

$$= \int \frac{1}{e^{x}} \cdot \frac{1}{x^2} (x+1) dx$$

$$= \int \frac{1}{e^{x}} \cdot \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$$

$$= \int \frac{1}{xe^{x}} dx + \int \frac{1}{e^{x}} \left(\frac{1}{x^2}\right) dx$$

$$= \int e^{-x} \left(\frac{1}{x}\right) dx + \int \frac{e^{-x}}{x^2} dx$$

$$= \int \frac{e^{-x}}{x}\,dx + \left[ e^{-x}\left(\frac{1}{x}\right) - \int \frac{-1}{x}(-e^{-x})\,dx \right]$$

$$= \int \frac{e^{-x}}{x}\,dx + \left[\frac{-e^{-x}}{x}\right] - \int \frac{e^{-x}}{x}\,dx$$

$$v = \frac{-e^{-x}}{x}$$

$$y_2 = v y_1 = \frac{-e^{-x}}{x} e^{x} = \frac{-1}{x}$$

$y_2 = \frac{-1}{x}$ is the soln of eqn ②

The general soln of eqn ②

$$y = c_1 e^{x} + c_2 (-1/x)$$

Let $y_1 = e^{x}$ and $y_2 = \frac{-1}{x} = -x^{-1}$

$$y_2' = -(-1\,x^{-1-1}) = +1\,x^{-2}$$

$$= \frac{+1}{x^2}$$

$y_1' = e^{x}$ & $y_2' = \frac{1}{x^2}$

$$w(y_1, y_2) = \begin{vmatrix} e^{x} & \frac{-1}{x} \\ e^{x} & 1/x^2 \end{vmatrix}$$

$$= e^{x}\cdot\frac{1}{x^2} + \frac{e^{x}}{x}$$

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$$= \frac{e^x}{x^2} + \frac{e^x}{x}$$

$$= e^x\left(\frac{1}{x^2} + \frac{1}{x}\right) = e^x\left(\frac{1+x}{x^2}\right)$$

$$v_1 = \int \frac{-R(x)\, y_2}{W(y_1, y_2)}\, dx$$

$$= \int \frac{-(x+1)}{e^x\left(\frac{1}{x^2} + \frac{1}{x}\right)} \cdot \left(-1/x\right) dx$$

$$= \int \frac{-(x+1)}{e^x\left(\frac{1+x}{x^2}\right)} \left(-1/x\right) dx$$

$$= \int \frac{1}{e^x} \left(\frac{x+1}{x}\right)\left(\frac{x^2}{(x+1)}\right) dx$$

$$= \int \frac{1}{e^x}\, x \cdot dx$$

$$v_1 = -xe^{-x} - e^{-x}$$

$$v_1 = -e^{-x}(x+1)$$

$$v_2 = \int \frac{R(x)\, y_1}{W(y_1, y_2)}$$

$$= \int \frac{(x+1)\, e^x}{e^x\left(1+x/x^2\right)} \cdot dx$$

$$= \int x^2 dx = \frac{x^3}{3}$$

$$v_2 = \frac{x^3}{2}$$

The Particular soln of eqn ①

$$y = v_1 y_1 + v_2 y_2$$

$$= -e^{-x}(x+1)\, e^{x} + \frac{x^3}{3}\left(-\frac{1}{x}\right)$$

$$y = -(x+1) - \frac{x^2}{3}$$

The general soln of the eqn ① is

$$y = c_1 e^{x} + c_2\left(\frac{-1}{x}\right) - (x+1) - \frac{x^2}{3}$$

Pb:-

$$y'' + y = \sec x . \tan x$$

Soln:-

$$y'' + y = \sec x . \tan x \quad \text{——} ①$$

corresponding to the homogenous eqn is

$$y'' + y = 0 \quad \text{——} ②$$

A.E is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$\therefore$ C.F is

$$y = e^{\alpha x}\left[c_1 \cos x + c_2 \sin x\right]$$

$y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x.$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$W = \cos^2 x + \sin^2 x = 1$$

$$v_1 = \int \frac{-R(x)\, y_2}{W(y_1, y_2)}\, dx$$

$$= \int \frac{-\sec x \cdot \tan x \cdot \sin x}{1} \cdot dx$$

$$= -\int \frac{1}{\cos x} \frac{\sin x}{\cos x} \cdot \sin x \cdot dx$$

$$= -\int \frac{\sin^2 x}{\cos^2 x}\, dx = -\int \tan^2 x\, dx$$

$$= \int -(\sec^2 x - 1)\, dx$$

$$= \int -\sec^2 x \cdot dx - \int dx$$

$$= -\tan x + x$$

$v_1 = x - \tan x$

$$v_2 = \int \frac{R(x)\, y_1}{y_1 y_2' - y_2 y_1'} \, dx$$

$$= \int \frac{\sec x \cdot \tan x}{1} \cdot \cos x \, dx$$

$$= \int \frac{\tan x}{1} \, dx \qquad = \int \frac{\sin x}{\cos x} \, dx$$

$v_2 = -\log \cos x \cdot dx$

$y = v_1 y_1 + v_2 y_2$

$y = (x - \tan x)\cos x + (-\log \cos x) \cdot \sin x$

$= x \cos x - \tan x \cdot \cos x - \log \cos x \cdot \sin x$

$y = x\,(\cos x - \sin x - \sin x \cdot \log \cos x)$ //

Unit - II

A review of Power Series:-

Explain power series and its convergence.

(i) An infinite series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x + \dots + a_n x^n + \dots$

is called a Power series in $x$.

(ii) $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is a power series in $(x - x_0)$

(iii) The series $\sum_{n=0}^{\infty} a_n x^n$ is said to be converges at a point $x$ if $\lim_{m \to \infty} \sum_{n=0}^{m} a_n x^n$ exits and in this case the sum of the series is the value of the limit

(iv) The arrangement of their pt of Convergence, all Power series in $x$ fall into one or another (or) three major catagories.

(i) $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2x^2 + 6x^3 + \dots$

(ii) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(iii) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$

The series (i) will diverges $\forall x \neq 0$

The series (ii) will converges $\forall x$

The series (iii) will converges for $|x| < 1$ and diverge for $|x| > 1$

Certain series of types converges for all values of $x$ in $|x| < R$ (R is radius of converges)

Suppose the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

i.e) $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$f''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots$

$f'''(x) = 6a_3 + 24 a_4 x + \cdots$

$f(0) = a_0$

$f'(0) = a_1 \Rightarrow a_1 = \dfrac{f'(0)}{1!}$

$f''(0) = 2a_2 \Rightarrow a_2 = \dfrac{f''(0)}{2!}$

Similarly,

$$a_3 = \frac{f'''(0)}{3!}, \text{ etc}$$

Now,

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^n(0)}{n!} x^n + \cdots$$

$$\therefore \quad a_n = \frac{f^n(0)}{n!}$$

Ratio test:-

Let $\sum_{n=0}^{\infty} a_n$ be a series of non-zero constants. Then we know that, if the limit

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$. exists, then the ratio test asserts that the series converges if $L < 1$. and diverges if $L > 1$.

In the case the power series $\sum_{n=0}^{\infty} a_n x^n$, we have

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

(1) use ratio test to verify that $R = 0$, $R = \infty$, $R = 1$ for the Series (i) $\sum_{n=0}^{\infty} n!\, x^n$

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(ii) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (iii) $\sum_{n=0}^{\infty} x^n$.

Soln:

(i) Consider the series

$$\sum_{n=0}^{\infty} n!\, x^n$$

we have $a_n = n!$

$a_{n+1} = (n+1)!$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n!}{n!\,(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{n+1} \right|$$

$$= 0$$

(ii) $a_n = \frac{1}{n!}$

$a_{n+1} = \frac{1}{(n+1)!}$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n!(n+1)}{n!} \right|$$

$$= \lim_{n\to\infty} |n+1|.$$

$= \infty$

(v) $a_n = 1$

$a_{n+1} = 1$

$$\therefore R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left( \frac{1}{1} \right)$$

$= 1.$

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Pb: (2) If $p$ is not zero or a +ve integer, show that the series

$$\sum_{n=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} x^n$$

converges for $|x|<1$ and diverges for $|x|>1$.

Soln:

Consider the series

$$\sum_{n=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} x^n$$

we have,

$$a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

$$a_{n+1} = \frac{p(p-1)(p-2)\cdots(p-n+1)(p-n)}{(n+1)!}$$

$$= \frac{(n+1)!}{n!(p-n)}$$

$$= \frac{n+1}{p-n}$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{p-n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n(1+1/n)}{n(p/n-1)} \right|$$

$$= \frac{1}{1} = 1$$

Hence the series converges for $|x|<1$ and diverges for $|x|>1$

Pb:③ we have $1+x+x^2+\dots+x^n = \dfrac{1-x^{n+1}}{1-x}$ of $x\neq 1$. we this formula to show $\dfrac{1}{1-x} = 1+x+x^2+\dots$ and

$\dfrac{1}{1+x} = 1-x+x^2-x^3+\dots$. Also show that $\log(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dots$

and $\tan^{-1}x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\dots$

Soln:

Given $1+x+x^2+\dots+x^n = \dfrac{1-x^{n+1}}{1-x}$ for $|x|<1$.

$$\lim_{n\to\infty} x^{n+1} = 0.$$

$$\lim_{n\to\infty} (1+x+x^2+\dots+x^n) = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x}$$

$$1+x+x^2+\dots = \frac{1}{1-x}$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+\dots \quad \text{—— (1)}$$

Replace $x$ by $-x$ in eqn (1)

$$\frac{1}{1-x} = 1-x+x^2-x^3+\dots \quad \text{—— (2)}$$

Integrate (2), we get

$$\int \frac{dx}{1+x} = \int (1-x+x^2-x^3+\dots)\,dx$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Replace $x$ by $x^2$ in (2), we get

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\dots$$

$$\int \frac{dx}{1+x^2} = \int (1-x^2+x^4-x^6+\dots)\,dx$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

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Ex. Find Power Series for $\frac{1}{(1-x)^2}$ from the series for $\frac{1}{1-x}$

(a) by squaring.

(b) by differenciating

Soln:

(a) we know that

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots \longrightarrow (i)$$

squaring (i), we get

$$\frac{1}{(1-x)^2} = (1+x+x^2+x^3+\cdots)^2$$

$$= (1+x+x^2+x^3+\cdots)(1+x+x^2+\cdots)$$

$$= (1+x+x^2+\cdots)(x+x^2+x^3+\cdots)$$

$$(x^2+x^3+x^4+\cdots)+\cdots$$

$$= 1+2x+3x^2+4x^3+\cdots$$

(b) Differentiating (i), we get

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$$

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## Power series solution for first order differential Equation

**Problem:**

(1) Find the Power series solution for the differential equation $y' = y$.

**Soln:**

Given $y' = y$ —— (1)

Assume (1) has a Power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n$$

(i.e) $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$

(1) $\Rightarrow a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

Equation the corresponding coefficients.

$a_1 = a_0 \Rightarrow a_1 = \frac{a_0}{1!}$

$2a_2 = a_1$

$2a_2 = a_0$

$a_2 = \frac{a_0}{2} = \frac{a_0}{2!}$

$3a_3 = a_2$

$3a_3 = \frac{a_0}{2!}$

$3a_3 = \frac{a_0}{2!\,3} = \frac{a_0}{3!}$

$4a_4 = a_3$

$a_4 = \frac{a_3}{4} = \frac{1}{4}\frac{a_0}{3!} = \frac{a_0}{4!}$

$\therefore$ The Power series solution of

(1) in

$$y = a_0 + \frac{a_0}{1!}x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \frac{a_0}{4!}x^4 + \dots$$

$$y = a_0\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]$$

i.e) $y = a_0 e^x$

Direct method

$$y' = y$$

$$\Rightarrow \frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{y} = dx$$

$$\Rightarrow \int \frac{dy}{y} = \int dx$$

$$\log y = x + \log c$$

$$\log y - \log c = x$$

$$\log (y/c) = x$$

$$\frac{y}{c} = e^{x}$$

$$\boxed{y = c . e^{x}}$$

Pb: Find the expression of $(1+x)^p$, where $p$ is arbitrary constant by using power series solution of differential equation (D.E)

Soln: let $y = (1+x)^p$

$$y' = p(1+x)^{p-1}(1)$$

$$(1+x)y' = p(1+x)^{p-1}(1+x)'$$

$$\Rightarrow (1+x)y' = p(1+x)^{p}$$

$$\Rightarrow (1+x)y' = py \quad \text{--- (1)}$$

$y'+xy'$

Also $y(0) = 1$. --- (2)

$y = (1+x)^p$ is a particular soln of the differential equation (1)

Assume that (1) has power series solution

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1} x^n + \dots$$

$$xy' = a_1 x + 2a_2 x^2 + 3a_3 x^3 + \dots + n a_n x^n + \dots$$

$$(1) \Rightarrow \overbrace{(a_1 + 2a_2 x + 3a_3 x^2 + \dots)}^{y'}$$

$$+ \overbrace{(a_1 x + 2a_2 x^2 + \dots)}^{xy'} = p\overbrace{(a_0 + a_1 x + a_2 x^2 + \dots)}^{y}$$

$$a_1 + (2a_2 + a_1)x + (3a_3 + 2a_2)x^2 + \dots$$

$$+ [(n+1)a_{n+1} + n a_n]x^n + \dots$$

$$= a_0 p + a_1 p x + a_2 p x^2 + \dots + p a_n x^n + \dots$$

$$\left[\, y(0) = a_0 \,,\quad y(0) = 1 \,\right]$$

$a_0 = 1.$

$a_1 = a_0\, P1 \Rightarrow \quad a_1 = 1 \cdot P$

$$a_1 = \frac{P}{1!}$$

$2a_2 + a_1 = a_1 P$

$2a_2 = a_1 P - a_1$

$= a_1\,(P-1)$

$$a_2 = \frac{a_1\,(P-1)}{2} = \frac{P\,(P-1)}{1!\,2}$$

$$a_2 = \frac{P\,(P-1)}{2!}$$

$3a_3 + 2a_2 = a_2 P$

$3a_3 = a_2 P - 2a_2$

$= a_2\,(P-2)$

$$a_3 = \frac{a_2\,(P-2)}{3}$$

$$a_3 = \frac{P(P-1)(P-2)}{3!}$$

$$a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

the Power Series soln of (1) is

$$y = 1 + \frac{px}{1!} + \frac{p(p-1)}{2!}x^2 + \cdots$$

$$+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} \rightarrow (3)$$

$\frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$

From (2) and (3)

$$(1+x)^p = 1 + \frac{p}{1!}x + \frac{p(p-1)}{2!}x^2$$

$$+ \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

$$+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots$$

---

Pbm Find a power series solution of the form $\sum a_n x^n$ of

(i) $y' + y = 1$ (ii) $xy' = y$

(iii) $y' = 2xy$. solve the equation directly, and Explain any discrepancies that arise.

Soln:-

(i) Given $y'+y=1 \longrightarrow ①$

Assume ① has a power series

solution

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

From ① $y'+y=1$

$(a_1 + 2a_2 x + 3a_3 x^2 + \ldots) + (a_0 + a_1 x + a_2 x^2 + \ldots)$
$= 1$

$(a_0 + a_1)(a_1 + 2a_2)x + (3a_3 + a_2)x^2$
$+ \ldots = 1 + 0x + 0x^2 + \ldots$

Comparing the coefficients

$a_0 + a_1 = 1$

$\boxed{a_1 = 1 - a_0}$ $\quad a_1 = -(a_0 - 1)$

$a_1 + 2a_2 = 0$

$2a_2 = -a_1$

$a_2 = \frac{-a_1}{2}$

$= \frac{-(1-a_0)}{2}$

$$a_2 = \frac{a_0 - 1}{2!}$$

$$3a_3 + a_2 = 0$$

$$3a_3 = -a_2$$

$$3a_3 = -\frac{(a_0-1)}{2}$$

$$a_3 = -\left(\frac{a_0-1}{3!}\right)$$

$$y = a_0 - \frac{(a_0-1)}{1!}x + \frac{a_0-1}{2!}x^2 - \frac{(a_0-1)}{3!}x^3 + \cdots$$

$$= (a_0 - 1 + 1) - \frac{(a_0-1)}{1!}x + \frac{(a_0-1)}{2!}x^2 - \frac{(a_0-1)}{3!}x^3 + \cdots$$

$$= [1 + (a_0-1)] - (a_0-1)x + \frac{(a_0-1)}{2!}x^2 - \frac{(a_0-1)}{3!}x^3 + \cdots$$

$$= 1 + (a_0-1)\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]$$

$$y = 1 + (a_0-1)e^{-x}$$

$$\Rightarrow y = 1 + ce^{-x} \quad \text{where } c = (a_0 - 1)$$

Direct method.

$y' + y = 1$

$\frac{dy}{dx} + y = 1$

$\frac{dy}{dx} = 1 - y$

$\frac{dy}{1-y} = dx$

$-\log(1-y) = x + c$

$\log(1-y)^{-1} = x + c$

$(1-y)^{-1} = e^{x} \cdot e^{c}$

$\frac{1}{1-y} = e^{x} \cdot c$

$1 = ce^{x}(1-y)$

$1 = ce^{x} - ce^{x}y$

$c \cdot e^{x} y = ce^{x} - 1$

$y = \frac{c \cdot e^{x} - 1}{c \cdot e^{x}} = \frac{\cancel{c \cdot e^{x}}}{\cancel{c \cdot e^{x}}} - \frac{1}{ce^{x}}$

$y = 1 - \frac{1}{ce^{x}} \Rightarrow y = 1 + c_1 e^{-x}$

where $c_1 = \frac{-1}{c}$

(ii) $xy' = y$

Soln:

Given $xy' = y \longrightarrow (1)$

Assume (1) has a power series solution.

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$

From (1)

$x(a_1 + 2a_2 x + 3a_2 x^2 + \dots)$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$\boxed{a_0 = 0}$

$\boxed{a_1 = a_1}$

$2a_2 = a_2$

$2a_2 - a_2 = 0$

$\boxed{a_2 = 0}$

$3a_3 - a_3 = 0$

$2a_3 = 0$

$\boxed{a_3 = 0}$

$y = 0 + a_1 x + 0 + \cdots$

$y = a_1 x$

Direct method

$xy' = y$

$x \dfrac{dy}{dx} = y$

$\dfrac{dy}{dx} = \dfrac{y}{x}$

$\dfrac{dy}{y} = \dfrac{dx}{x}$

$\log y = \log x + c \log c$

$\log y = \log cx$

$y = cx$

$y = a_1 x$ where $c = a_1$

(iii) $y' = 2xy$

Soln.

Given $y' = 2xy \longrightarrow (1)$

Assume (1) has power series

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

From (1),

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$= 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4$$

comparing co-efficients of $x$

$$\boxed{a_1 = 0}$$

$$2a_2 = 2a_0$$

$$\boxed{a_2 = a_0}$$

$$3a_3 - a_1 = 0$$

$$3a_3 = a_1$$

$$a_3 = a_1$$

$$\boxed{a_3 = 0}$$

$$4a_4 - 2a_2 = 0$$

$$4a_4 = 2a_2$$

$$a_4 = \frac{a_0}{2!}$$

From (1), we get

$$y = a_0 + \frac{a_1}{1!} x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \frac{a_4}{4!} x^4 + \cdots$$

$$y = a_0 + \frac{0}{1!}x + \frac{a_0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{a}{2!}x^4 \dots$$

$$y = a_0 + a_0 x^2 + \frac{a_0}{2!}x^3 + \dots$$

$$y = a_0 e^{x^2}$$

Direct method

$$y' = 2xy$$

$$\frac{dy}{dx} = 2xy$$

$$\frac{dy}{y} = 2x.dx$$

Integrating,

$$\log y. \; y = \frac{2x^2}{2} + c$$

$$y = ce^{x^2}$$

---

Pb: Express $\sin^{-1}x$ in the form of a Power series $\sum a_n x^n$ by solving $y' = (1-x^2)^{-1/2}$ in two ways. use this result to obtain the formula

$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 2^5}$

$+ \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 2^7} + \cdots$

Soln:-

Direct method,

$y' = (1-x^2)^{-1/2}$

$\frac{dy}{dx} = (1-x^2)^{-1/2}$

$\frac{dy}{dx} = \frac{1}{(1-x^2)^{1/2}}$

$\int \frac{dy}{dx} = \int \frac{1}{(1-x^2)^{1/2}}$ $\quad \therefore \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x$

$\int dy = \int \frac{dx}{(1-x^2)^{1/2}}$

$\int dy = \int \frac{dx}{\sqrt{1-x^2}}$

$y = \sin^{-1}(x) \longrightarrow (1)$

Assume (1) has power series expansion $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$

$(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots)$

$$= 1 + \frac{1}{2}(x^2) + \frac{(1/2)(1/2+1)}{2!} x^4$$

$$+ \frac{(1/2)(1/2+1)(1/2+2)}{3!} x^6 + \dots$$

Equating coefficients of $x$.

$\boxed{a_1 = 1}$

$2a_2 = 0$

$\boxed{a_2 = 0}$

$3a_3 = 1/2$

$\boxed{a_3 = 1/6}$

$4a_4 = 0$

$\boxed{a_4 = 0}$

$$5a_5 = \frac{(1/2)(3/2)}{2} = 3/8$$

$\boxed{a_5 = 3/40}$

$y = 0 + x + 0 + 1/6\, x^3 + 0 + 3/40\, x^5 + \dots$

$$y = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$$

$$\sin^{-1}x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots$$

put $x = 1/2$,

$$\sin^{-1}(1/2) = 1/2 + \frac{(1/2)^3}{6} + 3/40\,(1/2)^5$$

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

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## Second order differential equations and ordinary points.

Consider the differential equation $y'' + p(x)\,y' + Q(x)\,y = 0 \longrightarrow (1)$

Assume that $p(x)$ and $Q(x)$ are analytic at $x_0$.

Then each has a analytic at $x_0$. vaild in some nhbd of $x_0$.

In this case the point $x_0$ is called the Ordinary point of the equation (1).

And also every solution of (1) is analytic at $x_0$.

---

(1) Find Power Series Solution of $y''+y=0$ write down the general solution. of the form

$y = a_0 y_1(x) + a_1 y_2(x)$ where $y_1(x)$ and $y_2(x)$ are power series.

Soln.

Given $y''+y=0$ —— (1)

Here $P(x)=0$ and $Q(x)=1$

Clearly $P(x)$ and $Q(x)$ are analytic at all points.

Equation (1) has the power series solution of one from

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n \longrightarrow (2)$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

From (1),

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, \underline{x^n} + \sum_{n=0}^{\infty} a_n \underline{x^n} = 0$$

Equating the coefficients $x^n$ for $n = 0, 1, 2, \ldots$ separately to zero

$$(n+1)(n+2)\, a_{n+2} + a_n = 0.$$

$$(n+1)(n+2)\, a_{n+2} = -a_n$$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$$

put $n = 0$,

$$a_2 = \frac{-a_0}{2}$$

$$n = 1 \Rightarrow a_3 = \frac{-a_1}{6} = \frac{-a_1}{3!}$$

$$n = 2 \Rightarrow a_4 = \frac{-a_2}{12} = \frac{-(-a_0)}{2 \times 12} = \frac{+a_0}{24}$$

$$a_4 = \frac{+a_0}{4!}$$

$$a_4 = \frac{a_0}{4!}$$

When $n=3$,

$$a_5 = \cancel{\frac{a_0}{2}} \; \frac{-a_3}{4\times 5} = \frac{-(-a_1/3!)}{20}$$

$$= \frac{a_1}{20\times 6} = \frac{a_1}{5!}, \ldots$$

$$\therefore y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots$$

$$y = c_1 y_1 + c_2 y_2$$

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \quad (3)$$

This is the Power Series Solution of (1)

The Power Series in first term and Second term are of two solutions of equation (1)

clearly they are linearly.

independent

$\therefore$ (3) is a general solution

of (1).

(3) can be written as

$$y = a_0 \cos x + a_1 \sin x$$

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20/8/19 Solve: $(1+x^2)y'' + 2xy' - 2y = 0$

Soln,

$$(1+x^2)y'' + 2xy' - 2y = 0 \longrightarrow (1)$$

Here, $P(x) = \frac{2x}{1+x^2}$, $Q(x) = -\frac{2}{1+x^2}$

$P(x)$ and $Q(x)$ are analytic

at $x_0 = 0$. Let the power series

solution of (1) be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = 2 \cdot a_2 + 6a_3x + 12a_4x^2 + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1) \cdot a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \longrightarrow (2)$$

$$x^2y'' = 2a_2 \cancel{x^2} + 6a_3x^3 + 12a_4x^4 + \cdots$$

$$x^2(2a_2 + 6a_3x + 12a_4x^2 + \cdots$$

$$= x^2\left(\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}\right)$$

$$x^2y'' = \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} \longrightarrow (3)$$

$$2xy' = 2\sum_{n=0}^{\infty} n \cdot a_n x^n \longrightarrow (4)$$

$$-2y = -2\sum_{n=0}^{\infty} \cdot a_n x^n \longrightarrow (5)$$

② + ③ + ④ + ⑤ ,

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n$$

$$+ \sum_{n=0}^{\infty} n(n-1)\, \cancel{(n+2)(n+1)}\, a_n\, x^{n+2}$$

$$+ 2\sum_{n=0}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

equating the coefficients of $x^n$ to $0$.

$$(n+2)(n+1)\, a_{n+2} + n(n-1)\, a_n + 2n a_n - 2a_n = 0.$$

$$(n+2)(n+1)\, a_{n+2} = 2a_n - 2na_n - n(n-1)a_n$$

$n^2 - n$

$$= a_n\left[2 - 2n - n^2 + n\right]$$

$$= a_n(2 - n - n^2)$$

$$= a_n(-n^2 - n + 2)$$

$$a_{n+2} = \frac{a_n(-n^2 - n + 2)}{(n+2)(n+1)}$$

$$a_{n+2} = \frac{-a_n(n^2+n-2)}{(n+2)(n+1)}$$

$$= \frac{-a_n(n^2+2n-n-2)}{(n+2)(n+1)}$$

$$= \frac{-a_n \cancel{(n+2)}(\cancel{\quad})(n-1)}{(n+2)(\cancel{\quad})(n+1)}$$

$$a_{n+2} = -a_n\left(\frac{n-1}{n+1}\right)$$

When $n=0$, $a_2 = a_0$

$n=1 \Rightarrow a_3 = 0$

$n=2 \Rightarrow a_4 = -a_2(1/3) = \frac{-a_0}{3}$

$n=3 \Rightarrow a_5 = -a_3(2/4) = \cancel{\left(\frac{-a_0}{3}\right)(3/\ )}$

$= 0.$

$\frac{-a_0}{3}(3$

$n=4 \Rightarrow a_6 = -a_4\left(\frac{3}{5}\right) = \frac{a_0}{5}$

$n=5 \Rightarrow a_7 = -a_5(6(6) = 0$

$n=6 \Rightarrow a_8 = -a_6(5/7) = \frac{-a_0}{7}$

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$\therefore$ The Power Series Solution is

$$y = a_0 + a_1 x + a_0 x^2 + 0 + \left(-\frac{a_0}{3}\right) x^4$$

$$+ 0 + \frac{a_0}{5} x^6 + 0 + \cdots$$

$y = c_1 y_1 + c_2 y_2$

$$y = a_0 \left(1 + x^2 - \tfrac{1}{3} x^4 + \tfrac{1}{5} x^6 + \cdots\right)$$

$$+ a_1 x$$

Clearly these are independent solutions. $\therefore$ The general solution is,

$$y = a_0 \left(1 + x^2 - \tfrac{1}{3} x^4 + \tfrac{1}{5} x^6 + \cdots\right) + a_1 x$$

$$= a_0 \left[1 + x \tan^{-1} x\right] + a_1 x$$

21/8/17 **Theorem.**

Analytic

Let $x_0$ be an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0 \rightarrow (1)$. Let $a_0$, $a_1$ be the ordinary constants. Then there exists a unique function $y(x)$, that is analytic at $x_0$ is a solution of (1) in a certain neighborhood of this point and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Further more, if the power series expansion of $P(x)$ and $Q(x)$ are valid on the interval $|x - x_0| = R$, $R > 0$, then the Power series solution of this the expansion is also valid on the same interval.

* **Proof.** Given,

$$y'' + P(x)y' + Q(x)y = 0 \longrightarrow (2)$$

Let for convenience, $x_0 = 0$.

Then $P(x)$ and $Q(x)$ are analytic at $x_0$.

Now the power series expansion for $P(x)$ and $Q(x)$ are

$$P(x) = \sum_{n=0}^{\infty} P_n x^n$$

$$Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

that converges on the interval $|x| < R$. Now we seak the power series solution of this form of equation (2).

$$y = \sum_{n=0}^{\infty} a_n x^n \longrightarrow (3).$$

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$n = n+1$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \longrightarrow (4)$$

$$P(x)\, y' = \sum_{n=0}^{\infty} p_n x^n \left[ \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n \right]$$

$$= \sum_{n=0}^{\infty} x^n \left[ \sum_{k=0}^{n} p_{n-k} (k+1)\, a_{k+1} \right] \rightarrow (5)$$

[$\therefore$ by the result of Product of Power series].

$$Q(x)\, y = \left( \sum_{n=0}^{\infty} q_n x^n \right) \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} x^n \cdot \left( \sum_{k=0}^{n} a_k \cdot q_{n-k} \right) \rightarrow (6)$$

from (2), we get,

$$(4) + (5) + (6) = 0.$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} x^n + \sum_{n=0}^{\infty} x^n \left[ \sum_{k=0}^{n} p_{n-k} (k+1) a_{k+1} \right]$$

$$+ \sum_{n=0}^{\infty} x^n \left[ \sum_{k=0}^{n} a_k\, q_{n-k} \right] = 0$$

Equate the co-efficient of $x^n$ to zero

$$(n+1)(n+2)\, a_{n+2} + \sum_{k=0}^{n} p_{n-k} (k+1)\, a_{k+1}$$

$$+ \sum_{k=0}^{n} a_k\, q_{n-k} = 0.$$

$$(n+1)(n+2)\, a_{n+2} = -\sum_{k=0}^{n} \left[ p_{n-k}(k+1)\, q_{k+1} + a_k q_{n-k} \right]$$

$$\therefore\ a_{n+2} = \frac{-\sum_{k=0}^{n} \left[ p_{n-k}(k+1)\, q_{k+1} + a_k q_{n-k} \right]}{(n+1)(n+2)}$$

when $n=0$,

$$a_2 = \frac{-(p_0 a_1 + a_0 q_0)}{2}$$

when $n=1$,

$$a_3 = \frac{-\sum_{k=0}^{1} \left[ p_{1-k}\, (k+1)\, q_{k+1} + a_k q_{1-k} \right]}{3 \times 2}$$

$$= \frac{-(p_1 q_1 + a_0 q_1 + 2 p_0 a_2 + a_1 q_0)}{6}$$

when $n=2$,

$$a_4 = \frac{-\sum_{k=0}^{2} \left[ p_{2-k}\, (k+1)\, q_{k+1} + a_k \cdot q_{2-k} \right]}{3 \times 4}$$

$$= \frac{-\left[ p_2 a_1 + a_0 q_2 + p_1 (2) q_2 + a_1 q_1 + p_0 (3) a_3 + a_2 q_0 \right]}{12}$$

$\therefore$ Power series solution of (3).

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + \left(\frac{-(p_0 a_1 + a_0 q_1)}{2}\right) x^2$$

$$- \frac{(p_1 q_1 + a_0 q_1 + 2 p_0 q_2 + a_1 q_0)}{6} x^3$$

$$- \frac{(p_2 a_1 + a_2 q_2 + 2 p_1 a_2 + a_1 q_1 + 3 p_0 a_3 + a_2 q_0)}{12} x^4 + \cdots$$

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These formulas determine $a_2, a_3, \ldots$ in terms of $a_0$ and $a_1$.

So the resulting series satisfies equation (1) and the given interval condition is uniquely determine by these requirements.

<u>Note:-</u>

(i) The function $y_1(x)$ and $y_2(x)$ are infinite series for all non integrable values of P.

(ii) when $P$ is positive even integer $y_1(x)$ becomes a polynomial and if $P$ is positive integer $y_2(x)$ becomes a polynomial when $P=0$, this polynomial is $y_1(x)=1$.

when $P=1$, this polynomial is $y_2(x)=x$.

Similarly, if $P=2,3,4 \ldots$ the polynomials are $1-2x^2$, $x-2/3\,x^3$, $1-4x^2+2/3\,x^4 \ldots$ These are known as Hermitt's function.

Problem: 1

Consider the equation $y''+xy'+y=0$. Find its general solution in the form

$$y = a_0 y_1(x) + a_2 y_2(x).$$

Soln. Given,

$$y''+xy'+y=0 \longrightarrow (1)$$

Here $P(x) = x$, $Q(x) = 1$.

$P(x)$ and $Q(x)$ are analytic at a point $x=0$.

$$P(x) = \sum_{n=0}^{\infty} p_n x^n = x.$$

$$Q(x) = \sum_{n=0}^{\infty} q_n x^n = 1.$$

That are converges in the interval $|x| < R$.

Now, we seek the power series of the equation ① in the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$y' = \sum_{n=0}^{\infty} n \cdot a_n x^{n-1}$$

$$P(x) y' = xy' = \sum_{n=0}^{\infty} n \cdot a_n x^n$$

$$y'' = 2a_2 + 6a_3 + 12 a_4 x^2 + \ldots$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) x^n a_{n+2}$$

$$① \Rightarrow \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n$$

$$+ \sum_{n=0}^{\infty} n \cdot a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Equate the coefficients of $x^n = 0$.

$$(n+1)(n+2)\, a_{n+2} + n a_n + a_n = 0$$

$$(n+1)(n+2)\, a_{n+2} = -(n+1)\, a_n$$

$$\therefore a_{n+2} = \frac{-a_n}{n+2}$$

When $n=0$, $a_2 = \dfrac{-a_0}{2}$

when $n=1$, $a_3 = \dfrac{-a_1}{3}$

when $n=2$, $a_4 = \dfrac{-a_2}{4} = \dfrac{a_0/2}{4} = \dfrac{a_0}{8}$

when $n=3$, $a_5 = \dfrac{-a_3}{5} = \dfrac{a_1/3}{5} = \dfrac{a_1}{15}$

when $n=4$, $a_6 = \dfrac{-a_4}{6} = \dfrac{-a_0/8}{6} = \dfrac{-a_0}{48}$

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$\therefore y = a_0 + a_1 x + a_2 x^2 + \cdots$

$$= a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{3} x^3 + \frac{a_0}{8} x^4$$

$$+ \frac{a_1}{15} x^5 - \frac{a_0}{48} x^6 + \cdots$$

i.e) $y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{15} + \cdots\right)$

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The equation $y'' + \left(p + \frac{1}{2} - \frac{1}{4}x^2\right)y = 0$, where $p$ is a constant, certainly has a series solution of the form $y = \sum a_n x^n$.

(a) show that the coefficients $a_n$ are related by the three form recursion formula.

$$(n+1)(n+2)a_{n+2} + \left(p + \tfrac{1}{2}\right)a_n - \frac{1}{4}a_{n-2} = 0.$$

(b) If the depedent variable in changed from $y$ to $w$ by means of $y = w.e^{-x^2/4}$, show that the equation is transform into $w'' - xw' + pw = 0$.

(c) verify that the equation in (b) has a two term recursion formula and find its general solution.

Soln:-

(a) Given

$$y'' + (p + 1/2 - \frac{1}{4}x^2)y = 0 \longrightarrow (1)$$

i.e) $-y'' + (p+1/2)y - \frac{1}{4}x^2 y = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be a power series solution of the equation (1)

$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$(p+1/2)\,y = (p+1/2)\sum_{n=0}^{\infty} a_n x^n$$

$$-\frac{1}{4}x^2 y = -\frac{1}{4}\sum_{n=0}^{\infty} a_n x^n x^2$$

$$= -\frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= -\frac{1}{4}\sum_{n=2}^{\infty} a_{n-2} x^n$$

$$(4) \Rightarrow \sum_{n=0}^{\infty}(n+1)(n+2) \quad x^n + (p+1/2)\sum_{n=0}^{\infty} a_n x^n$$

$$-\frac{1}{4}\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Equate the coefficients of $x^n = 0$.

$$(n+1)(n+2)\,a_{n+2} + (p+1/2)\,a_n - \frac{1}{4}a_{n-2} = 0.$$

(b) Let $y = we^{-x^2/4}$

$$y' = w' e^{-x^2/4} + w.e^{-x^2/4}\left(-\frac{2x}{4}\right)$$

$$y' = w' e^{-x^2/4} - \frac{1}{2} wx e^{-x^2/4}$$

$$= we^{-x^2/4} - 1/2\, wxe^{-x^2/4}$$

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$$y'' = w'' e^{-x^2/4} + w' e^{-x^2/4}\left(-\frac{2x}{4}\right)$$

$$-\frac{1}{2} w' x e^{-x^2/4} - \frac{1}{2} w e^{-x^2/4} - \frac{1}{2} w x e^{-x^2/4}\left(\frac{-2x}{4}\right)$$

$$= e^{-x^2/4}\left[w'' - \frac{x}{2} w' - \frac{1}{2} w' x - \frac{1}{2} w + \frac{x}{4} w\right]$$

$$y'' = e^{-x^2/4}\left[w'' - xw' - \frac{1}{2} w + \frac{x^2}{4} w\right]$$

(1) $\Rightarrow$

$$e^{-x^2/4}\left(w'' - xw' - \frac{1}{2} w + \frac{x^2}{4} w\right)$$

$$+(p + 1/2) w e^{-x^2/4} - x^2/4 \, w e^{-x^2/4} = 0$$

÷ by $e^{-x^2/4}$

$$w'' - xw' - \cancel{1/2 w} + \cancel{x^2/4 w} + pw + \cancel{1/2 w}$$

$$\cancel{-x^2/4 w} = 0.$$

$$w'' - xw' + pw = 0 \longrightarrow (2).$$

(c) Let $w = \sum_{n=0}^{\infty} a_n x^n$ be the power

Series solution of (2).

$$w' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \therefore xw' = \sum_{n=1}^{\infty} n a_n x^n$$

$$w'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$w'' - xw' + pw = 0$$

(2) $\Rightarrow$ $\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$$- \sum_{n=1}^{\infty} n a_n x^n + p \sum_{n=0}^{\infty} a_n x^n = 0.$$

Equate the coefficients of $x^n = 0$.

$$(n+1)(n+2) a_{n+2} - n a_n + p a_n = 0$$

$$(n+1)(n+2) a_{n+2} - (n-p) a_n = 0.$$

$$\therefore a_{n+2} = \frac{(n-p)}{(n+1)(n+2)} a_n$$

$n=0 \Rightarrow$ $\boxed{a_2 = \frac{-p}{2} a_0}$

$n=1 \Rightarrow$ $a_3 = \frac{(1-p)}{6} a_1 = \frac{-(p-1)}{3!} a_1$

$$n=2 \Rightarrow a_4 = \frac{2-p}{12} a_2$$

$$= \frac{-(p-2)}{12}\left(\frac{-p}{2} a_0\right)$$

$$\boxed{a_4 = \frac{(p-2)p}{4!} a_0}$$

$$n=3 \Rightarrow a_5 = \frac{3-p}{4.5} a_3$$

$$= \frac{-(p-3)}{4.5}\left(\frac{-(p-1)}{3!} a_1\right)$$

$$\boxed{a_5 = \frac{(p-1)(p-3)}{5!} a_1}$$

$$n=4 \Rightarrow a_6 = \frac{4-p}{5.6} a_4$$

$$= \frac{-(p-4)}{5.6}\left(\frac{(p-2)p}{4!} a_0\right)$$

$$\boxed{a_6 = \frac{-p(p-2)(p-4)}{6!} a_0}$$

The Power Series solution of

(2) in $w = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

i.e) $w = a_0 + a_1 x - \frac{p}{2!} a_0 x^2$

$$- \frac{(p-1)}{3!} a_1 x^3 + \frac{p(p-2)}{4!} a_0 x^4$$

$$+ \frac{(p-1)(p-3)}{5!} a_1 x^5 - \frac{p(p-2)(p-4)}{6!} a_0 x^6 + \cdots$$

$$= a_0 \left[ 1 - \frac{p}{2!} x^2 + \frac{p(p-2)}{4} x^4 - \frac{p(p-2)(p-4)}{6!} x^6 + \cdots \right.$$

$$\left. + a_1 \left[ x - \frac{(p-1)}{3!} x^3 + \frac{(p-1)(p-3)}{5!} x^5 + \cdots \right] \right.$$

$w = a_0 y_1(x) + a_1 y_2(x) \longrightarrow (3)$

~~th~~ here $y_1(x)$ and $y_2(x)$ are linearly indepedent.

$\therefore$ (3) is the general solution of (2).

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~~cheb~~ chebyshev's equation is

$(1-x^2) y'' - xy' + p^2 y = 0$ where $p$ is a constant

(a) Find two linearly independent

Solutions valid for $|x|<1$.

(b) show that if $p = n$ where $n$ is an integer $\geq 0$, then there is a polynomial solution of degree $n$. when these are multiplied by suitable constants, they are called the chebyshev polynomials.

Soln.

Given,

$$(1-x^2)y'' - xy' + p^2 y = 0 \longrightarrow (1)$$

let $y = \sum_{n=0}^{\infty} a_n x^n$ be the Power Series Solution of (1).

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$xy' = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$x^2 y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n$$

$$\therefore \cancel{x^2 y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^{n+2}}$$

Then eqn ① becomes,

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n - \sum_{n=0}^{\infty} n(n-1)\, a_n x^n$$

$$- \sum_{n=0}^{\infty} n a_n x^n + p^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Equate the coefficient of $x^n = 0$

$$(n+1)(n+2)\, a_{n+2} - n(n-1) a_n - n a_n + p^2 a_n = 0.$$

$$(n+1)(n+2)\, a_{n+2} = n(n-1) a_n + n a_n - p^2 a_n$$

$$\therefore a_{n+2} = \frac{(n(n-1) + n - p^2)\, a_n}{(n+1)(n+2)}$$

$$\therefore a_{n+2} = \frac{(n^2 - p^2)}{(n+1)(n+2)}\, a_n$$

when $n = 0 \Rightarrow a_2 = \frac{-p^2}{2} a_0$

When $n=1$,

$$a_3 = \frac{(1-p^2)}{6} a_1 = \frac{(1-p^2)}{3!} a_1 = \frac{-(p^2-1)}{3!} a_1$$

$$n=2, \Rightarrow a_4 = \frac{(4-p^2)}{12} a_2 = \frac{-(p^2-4)}{4\cdot 3}\left(\frac{-p^2}{2} a_0\right)$$

$$\cancel{n=3} \Rightarrow \cancel{a_5} = \frac{p^2(p^2-4)}{4!} a_0$$

$$a_4 = \frac{p^2(p^2-2^2)}{4!} a_0$$

$$n=3 \Rightarrow a_5 = \frac{(3^2-p^2)}{4\times 5} a_3$$

$$= \frac{(3^2-p^2)}{5\times 4}\left(\frac{-(p^2-1)}{3!} a_1\right)$$

$$a_5 = \frac{(p^2-1^2)(p^2-3^2)}{5!} a_1$$

$$n=4 \Rightarrow a_6 = \frac{(4^2-p^2)}{5\times 6} a_4$$

$$= \frac{-(p^2-4^2)}{5\times 6}\left(\frac{p^2(p^2-2^2)}{4!} a_0\right) \dots$$

$$a_6 = \frac{-p^2(p^2-2^2)(p^2-4^2)}{6!} a_0$$

when $n=5$,

$$a_7 = \frac{5^2-p^2}{6\times 7} a_5$$

$$= \frac{-(p^2-5^2)}{6\times 7}\left(\frac{(p^2-1^2)(p^2-3^2)}{5!} a_1\right)$$

$$a_7 = -\frac{(p^2-1^2)(p^2-3^2)(p^2-5^2)}{7!} a_1, \ldots$$

$$y = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$y = a_0 + a_1 x - \frac{p^2}{2} a_0 x^2 - \frac{(p^2-1)}{3!} a_1 x^3$$

$$+ \frac{p^2(p^2-2^2)}{4!} a_0 x^4 + \frac{(p^2-1^2)(p^2-3^2)}{5!} a_1 x^5$$

$$- \frac{p^2(p^2-2^2)(p^2-4^2)}{6!} a_0 x^6 -$$

$$\frac{(p^2-1^2)(p^2-3^2)(p^2-5^2)}{7!} a_1 x^7 + \ldots$$

$$y = a_0\left[1 - \frac{p^2}{2!}x^2 + \frac{p^2(p^2-2^2)}{4!}x^4 - \frac{p^2(p^2-2^2)(p^2-4^2)}{6!}x^6 + \cdots\right]$$

$$+ a_1\left[x - \frac{(p^2-1)}{3!}x^3 + \frac{(p^2-1^2)(p^2-3^2)}{5!}x^5 - \frac{(p^2-1^2)(p^2-3^2)(p^2-5^3)}{7!}x^7 + \cdots\right]$$

$$y = a_0\, y_1(x) + a_1\, y_2(x)$$

$p = 0$, $y_1(x) = 1$

$p = 1$, $y_2(x) = x$

$p = 2$, $y_1(x) = 1 - 2x^2$

$p = 3$, $y_2(x) = x - 4/3\, x^3$

$p = 4$, $y_1(x) = x - 2x^2 + 8x^4 + \cdots$

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Problem:

V.Q

verify that the equation $y'' + y' - xy = 0$ has the three term recursion formula and find its Series soln $y_1(x)$ and $y_2(x)$ such that

(a) $y_1(0) = 1, \quad y_1'(0) = 0$

(b) $y_2(0) = 0, \quad y_2'(0) = 1$

Soln.

$$y'' + y' - xy = 0 \longrightarrow (1)$$

let $y = \sum_{n=0}^{\infty} a_1 x^n$ be a power Series

solution by equation (1),

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \longrightarrow (2)$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} a_{n-1} x^n$$

$$y'' + y' - xy = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n$$

$$- \sum_{n=0}^{\infty} a_{n-1}\, x^n = 0 .$$

Clearly the coefficients of $x^n$ for seperately zero, $n = 0, 1, 2, \ldots$

$$(n+1)(n+2)\, a_{n+2} + (n+1)\, a_{n+1} - a_{n-1} = 0 .$$

$$\therefore \quad a_{n+2} = \frac{a_{n-1} - (n+1)\, a_{n+1}}{(n+1)(n+2)}$$

when $n=0$,

$$a_2 = \frac{-a_1}{2!}$$

when $n=1$,

$$a_3 = \frac{a_0 + a_1}{3!}$$

when $n=2$,

$$a_4 = \frac{a_1 - a_0}{4!}$$

when $n=3$,

$$a_5 = \frac{-4a_1 + a_0}{5!}$$

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$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x - \frac{a_1}{2!} x^2 + \left(\frac{a_0 + a_1}{3!}\right) x^3$$

$$+ \left(\frac{a_1 - a_0}{4!}\right) x^4 + \left(\frac{4a_1 + a_0}{5!}\right) x^5 + \cdots$$

$$y = a_0 \left[1 + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots\right]$$

$$+ a_1 \left[x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} + \cdots\right]$$

$$y_1(x) = 1 + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$$

$$y_2(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} + \cdots$$

$$y_1(0) = 1, \quad y_2(0) = 0.$$

$$y_1'(x) = \frac{3x^2}{3!} - \frac{4x^3}{4!} + \frac{5x^4}{5!} - \cdots$$

$$y_2'(x) = 1 - \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} - \frac{20x^4}{5!} + \cdots$$

$$y_2'(0) = 1, \quad y_1'(0) = 0.$$

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# Unit - III

## Regular Singular Points

**Defn:**

diff. (not analytic) exists → singular point

A point $x_0$ is called singular point of differential equation $y'' + P(x)y' + Q(x)y = 0$, if one or other or both of the coefficients of functions $P(x)$ and $Q(x)$ fails to analytic at $x_0$.

**example**

Consider, $x^2 y'' + 2xy' - 2y = 0$.

$$y'' + \frac{2xy'}{x^2} - \frac{2y}{x^2} = 0.$$

$$y'' + \frac{2}{x} y' - \frac{2}{x^2} y = 0.$$

Here $P(x) = \frac{2}{x}$, $Q(x) = -\frac{2}{x^2}$

put $x = 0$,

$$P(x) = \frac{2}{0} = \infty, \quad Q(x) = \frac{-2}{0} = \infty$$

$P(x)$ and $Q(x)$ are not analytic at zero.

$\therefore x=0$ is a Singular point

Defn

A Singular point $x_0$ of differential equation $y'' + p(x)y' + Q(x)(y) = 0$ is said to be a regular if $(x-x_0) P(x)$ and $(x-x_0)^2 Q(x)$ are analytic otherwise the Singular point is irregular.

Example

Consider, $x^2y'' + 2xy' - 2y = 0$

$$y'' + \frac{2xy'}{x^2} - \frac{2y}{x^2} = 0$$

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0.$$

$P(x) = \frac{2}{x}$, $Q(x) = -\frac{2}{x^2}$

$P(x)$ and $Q(x)$ are not analytic at $x=0$.

Hence $x=0$ is a Singular point

$x\,p(x) = x\left(\frac{2}{x}\right) = 2$

$x^2\,Q(x) = x^2\left(-\frac{2}{x^2}\right) = -2$

Here $x\,p(x)$ and $x^2\,Q(x)$ are analytic at $x=0$.

$\therefore$ The origin is regular singular point.

Example.

Consider the Legendre polynomial

$(1-x^2)y'' - 2xy' + p(p+1)y = 0$

$$y'' - \frac{2xy'}{1-x^2} + \frac{p(p+1)y}{1-x^2} = 0.$$

Here, $P(x) = \frac{-2x}{1-x^2}$, $Q(x) = \frac{p(p+1)}{1-x^2}$

$P(\pm 1) = \infty$, $Q(\pm 1) = \infty$.

$\therefore$ $P(x)$ and $Q(x)$ are not analytic at $x = \pm 1$.

$\therefore$ $x = \pm 1$ are singular points

Take $x_0 = 1$

$$(x-x_0)\,P(x) = (x-1)\left(\frac{-2x}{1-x^2}\right)$$

$$= \frac{2x}{1+x}$$

$$(x-x_0)^2\,Q(x) = (x-1)^2\,\frac{p(p+1)}{1-x^2}$$

$$= -\frac{(x-1)\,p(p+1)}{1+x}$$

Here $(x-1)P(x)$ and $(x-1)^2 Q(x)$ are analytic at $x=1$.

$\therefore$ $x=1$ is a regular singular point

Take $x_0 = -1$

$$(x-x_0)\,P(x) = (x+1)\left(\frac{-2x}{1-x^2}\right)$$

$$= \frac{-2x}{1-x}$$

$$(x-x_0)^2\,Q(x) = \frac{(x+1)^2\,p(p+1)}{1-x^2}$$

$$= \frac{(x+1)\,p(p+1)}{1-x}$$

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Here $(x-x_0)p(x)$ and

$(x-x_0)^2 Q(x)$ are analytic at $x=-1$

$\therefore x=-1$ is a regular singular

point

Problem

① Determine the nature of

Singular points of

$x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$.

$$y'' - \frac{2(x-1)}{x^3(x-1)}y' + \frac{3x}{x^3(x-1)}y = 0.$$

$$y'' - \frac{2}{x^3}y' + \frac{3}{x^2(x-1)}y = 0$$

Here $p(x) = \frac{-2}{x^3}$, $Q(x) = \frac{3}{x^2(x-1)}$

$p(x)$ and $Q(x)$ are not analytic

at $x=0,1$

$\therefore x=0,1$ are singular points.

Take, $x_0 = 0$

$$(x-x_0)\,p(x) = (x-0)\left(\frac{-2}{x^3}\right) = \frac{-2}{x^2}$$

$$(x-x_0)^2\,Q(x) = (x-0)^2\left(\frac{3}{x^2(x-1)}\right) = \frac{3}{x-1}$$

$\therefore$ $(x-x_0)\,p(x)$ is not analytic at $x=0$.

$\therefore$ But $(x-x_0)^2\,Q(x)$ is analytic at $x=0$.

Hence $x=0$ is a irregular singular point

Take $x=1$,

$$(x-x_0)\,p(x) = (x-1)\left(\frac{-2}{x^3}\right)$$

$$(x-x_0)^2\,Q(x) = (x-1)^2\left(\frac{3}{x^2(x-1)}\right) = \frac{3(x-1)}{x^2}$$

$\therefore$ $(x-x_0)\,p(x)$ and $(x-x_0)^2\,Q(x)$ are analytic at $x=1$

$\therefore$ $x=1$ is a regular singular point

Here $(x-x_0)P(x)$ and

$(x-x_0)^2 Q(x)$ are analytic at $x=-1$

$\therefore x=-1$ is a regular singular

point

Problem

① Determine the nature of

Singular points of

$$x^3(x-1)y'' - 2(x-1)y' + 3xy = 0.$$

$$y'' - \frac{2(x-1)}{x^3(x-1)}y' + \frac{3x}{x^3(x-1)}y = 0.$$

$$y'' - \frac{2}{x^3}y' + \frac{3}{x^2(x-1)}y = 0$$

Here $P(x) = \frac{-2}{x^3}$, $Q(x) = \frac{3}{x^2(x-1)}$

$P(x)$ and $Q(x)$ are not analytic

at $x=0,1$

$\therefore x=0,1$ are singular points.

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Take, $x_0 = 0$

$$(x-x_0)\,P(x) = (x-0)\left(\frac{-2}{x^3}\right) = \frac{-2}{x^2}$$

$$(x-x_0)^2\,Q(x) = (x-0)^2\left(\frac{3}{x^2(x-1)}\right) = \frac{3}{x-1}$$

$\therefore$ $(x-x_0)\,P(x)$ is not analytic at $x=0$.

$\therefore$ But $(x-x_0)^2\,Q(x)$ is analytic at $x=0$.

Hence $x=0$ is a irregular singular point

Take $x=1$,

$$(x-x_0)\,P(x) = (x-1)\left(\frac{-2}{x^3}\right)$$

$$(x-x_0)^2\,Q(x) = (x-1)^2\left(\frac{3}{x^2(x-1)}\right) = \frac{3(x-1)}{x^2}$$

$(x-x_0)\,P(x)$ and $(x-x_0)^2\,Q(x)$ are analytic at $x=1$

$\therefore$ $x=1$ is a regular singular point

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(20). Determine the nature of singular point of

$$x^2(x^2-1)^2 y'' - x(1-x)y' + 2y = 0.$$

Solution:

$$y'' - \frac{x(1-x)}{x^2(x^2-1)} y' + \frac{2y}{x^2(x^2-1)^2} = 0$$

$$P(x) = \frac{-(1-x)}{x(x-1)^2(x+1)^2}$$

$$= \frac{1}{x(x+1)^2(x-1)}$$

$$Q(x) = \frac{2}{x^2(x^2-1)^2}$$

$$= \frac{2}{x^2(x-1)^2(x+1)^2}$$

$P(x)$ and $Q(x)$ are not analytic at $x=0, \pm 1$

Hence $x=0, \pm 1$ are singular points

Take $x_0 = 0$

$$(x-x_0)\, P(x) = x \frac{1}{x(x+1)^2(x-1)} = \frac{1}{(x+1)^2(x-1)}$$

$$(x-x_0)^2\, Q(x) = x^2 \frac{2}{x^2(x^2-1)^2} = \frac{2}{(x^2-1)^2}$$

$(x-x_0)P(x)$, $(x-x_0)^2 Q(x)$ are analytic at $x_0=0$

$\therefore$ $x=0$ is regular Singular point.

Take $x=1$

$$(x-1)\, p(x) = (x-1) \frac{1}{x(x+1)^2(x-1)} = \frac{1}{x(x+1)^2}$$

$$(x-1)^2\, Q(x) = \frac{2}{x^2(x+1)^2}$$

$(x-1)\, p(x)$, $(x-1)^2 Q(x)$ are analytic at $x=1$.

$\therefore$ $x=1$ is a regular Singular point.

Take $x=-1$

$$(x+1)\, P(x) = \frac{1}{x(x+1)(x-1)}$$

$$(x+1)^2\, Q(x) = \frac{2}{x^2(x-1)^2}$$

$(x+1)\,p(x)$ is not analytic at $x=-1$

$\therefore$ $x=-1$ is an irregular Singular point. /r.

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③. Determine the natural of point $x=0$ for the following equation

(a) $y'' + (\sin x) y = 0$  b) $x^3 y'' + (\sin x) y = 0$

c) $x y'' + (\sin x) y = 0$  d.) $x^2 y'' + \sin x\, y = 0$.

Soln.

(a) Given $y'' + \sin x\, y = 0$.

$p(x) = 0$, $Q(x) = \sin x$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$p(x)$ and $Q(x)$ are analytic at $x=0$

$x=0$ is not Singular point

(b) Given, $x^3 y'' + (\sin x) y = 0$.

$$y'' + \frac{\sin x}{x^3} y = 0.$$

$p(x) = 0$ and $Q(x) = \dfrac{\sin x}{x^3}$

$$= \frac{1}{x^3}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$\therefore \boxed{x=0}$

$P(x)$ and $Q(x)$ is not analytic at $x=0$.

Hence $x=0$ is singular point.

put $x_0=0$.

$$(x-x_0)P(x) = (x-0)\cdot 0 = 0.$$

$$(x-x_0)^2 \cdot Q(x) = \frac{x^2}{x^3}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$$

$(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are analytic at $x=0$.

Hence $x=0$ is regular singular point

Smaira

(Q) Given, $xy'' + (\sin x)y = 0$.

$$y'' + \frac{\sin x}{x}\cdot y = 0.$$

$$P(x) = 0$$

$$Q(x) = \frac{\sin x}{x}$$

$$= \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{x}{x}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right]$$

$$= 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots$$

$P(x)$ and $Q(x)$ is analytic at $x=0$.

Hence $x=0$ is regular singular point

(d) $x^4 y'' + \sin x \, y = 0$.

$$y'' + \frac{\sin x}{x^4} y = 0$$

$P(x) = 0$ and $Q(x) = \frac{\sin x}{x^4}$

$$Q(x) = \frac{1}{x^4}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right]$$

$$= \frac{x}{x^4}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right]$$

$$= \frac{1}{x^3}-\frac{1}{x\,3!}+\frac{x}{5!}-\cdots$$

$P(x)$ and $Q(x)$ is not analytic at $x=0$.

Here $x=0$ is singular point put $x_0 = 0$

$(x-x_0)\,P(x) = x_0 = 0$

$$(x-x_0)^2\, Q(x) = \frac{x^2}{x^4}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right]$$

$$= \frac{1}{x}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \frac{1}{x} - \frac{x}{3!} + \frac{x^3}{5!} \cdots$$

$(x-x_0)\,P(x)$ and $Q(x)$ are analytic at $x=0$

But $(x-x_0)^2 Q(x)$ is not analytic at $x=0$

Hence $x=0$ is irregular singular point.

i.e) $x^2 y'' + (\sin x)\, y = 0$.

$$y'' + \frac{\sin x}{x^2}\, y = 0$$

$$P(x) = 0, \quad Q(x) = \frac{\sin x}{x^2}$$

$$Q(x) = \frac{1}{x^2}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{x}{x^2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right] = \frac{1}{x} - \frac{x}{3!} + \frac{x^3}{5!} \cdots$$

$P(x)$ and $Q(x)$ are is not analytic at $x=0$.

Hence $x=0$ is a singular point

put $x=0$

$$(x-x_0)\,P(x) = x \cdot 0 = 0,$$

$$(x-x_0)^2 Q(x) = x^2\left[\frac{1}{x} - \frac{x}{3!} + \frac{x^3}{5!} - \cdots\right]$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$P(x)$, $Q(x)$ are analytic at $x=0$.

Hence $x=0$ is a ~~irg~~ regular singular point

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(4) For each of the following differential equation locate and classify the singular points.

(a) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$

(b) $(3x+1)xy'' - (x+1)y' + 2y = 0$.

(c) $x^2y'' + (2-x)y' = 0$.

Soln:

Given,

(a) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$

$\div$ by $x^3(x-1)$

$$y'' - \frac{2(x-1)}{x^3(x-1)}y' + \frac{3x}{x^3(x-1)}y = 0$$

$$y'' - \frac{2}{x^3}y' + \frac{3}{x^2(x-1)}y = 0$$

$$P(x) = \frac{-2}{x^3} \quad ; \quad Q(x) = \frac{3}{x^2(x-1)}$$

$P(x)$ is not analytic at $x=0$

$Q(x)$ is not analytic at $x=0,1$

$\therefore$ $x=0,1$ are singular points of (1)

$x\,p(x) = -\frac{2}{x^3}$ ; $x^2\,Q(x) = \frac{3}{x-1}$

$x\,p(x)$ is not analytic at $x=0$.

$\therefore x=0$ is an irregular Singular point of (1)

$$(x-1)\,p(x) = -\frac{2(x-1)}{x^3},$$

$$(x-1)^2\,Q(x) = \frac{3(x-1)}{x^2}$$

$(x-1)\,p(x)$ and $(x-1)^2\,Q(x)$ are analytic at $x=1$

$\therefore x=1$ is a regular Singular point of (1).

(b) $(3x+1)\,x y'' - (x+1)\,y' + 2y = 0$

$\div$ by $(3x+1)x$

$$y'' - \frac{(x+1)}{(3x+1)x}\,y' + \frac{2}{(3x+1)\,x}\,y = 0$$

$$p(x) = -\frac{(x+1)}{(3x+1)x}, \quad Q(x) = \frac{2}{(3x+1)\,x}$$

$P(x)$ and $Q(x)$ are not analytic at $x = 0, -1/3$

$x = 0, -1/3$ are singular points

$$x\,P(x) = \frac{-x(x+1)}{x(3x+1)} = \frac{-(x+1)}{(3x+1)}$$

$$x^2 Q(x) = \frac{2x}{x(3x+1)} = \frac{2}{x(3x+1)}$$

$x\,P(x)$ and $x^2 Q(x)$ are analytic at $x = 0$.

$\therefore$ $x = 0$ is a irregular singular point.

$$(3x+1)\,P(x) = \frac{-(x+1)(3x+1)}{x(3x+1)} = \frac{-(x+1)}{x}$$

$$(3x+1)^2\,Q(x) = \frac{2(3x+1)^2}{x(3x+1)} = \frac{2(3x+1)}{x}$$

$(3x+1)\,P(x)$, $(3x+1)^2\,Q(x)$ are analytic at $x = -1/3$

$\therefore$ $x = -1/3$ is a regular singular point

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(6). $x^2 y'' + (2-x) y' = 0$

÷ by $x^2$

$$y'' + \frac{2-x}{x^2} y' = 0$$

$p(x) = \frac{2-x}{x^2}$ ; $Q(x) = 0$.

$p(x)$ is not analytic at $x=0$.

$\therefore x=0$ is a singular point

$xp(x) = \frac{2-x}{x}$ , $x^2 Q(x) = 0$.

$xp(x)$ is not analytic at $x=0$.

$\therefore x=0$ is an irregular singular point.

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(5). Consider the differential equation

$$y'' + \frac{1}{x^2} y' - \frac{1}{x^3} y = 0.$$

(a) show that $x=0$ is an irregular singular point

(b) use the fact that $\boxed{y_1 = x}$ is a solution to find a second independent solution $y_2$:

Soln. Given

(a) $$y'' + \frac{1}{x^2} y' - \frac{1}{x^3} y = 0$$

$$p(x) = \frac{1}{x^2} \; ; \quad Q(x) = \frac{1}{x^3}$$

$p(x)$, $Q(x)$ are not analytic at $x=0$.

$\therefore$ $x=0$ is a Singular point.

$$x p(x) = \frac{1}{x} \; , \quad x^2 Q(x) = \frac{-1}{x}$$

$xp(x)$, $x^2 Q(x)$ are not analytic at $x=0$.

$\therefore$ $x=0$ is an irregular Singular point

(b) Let $y_1 = x$ be a solution of (1).

Let $\boxed{y_2 = v y_1}$

now, $v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}$

$$= \int \frac{1}{x^2} e^{-\int \frac{1}{x^2} dx} dx$$

$$= \int \frac{1}{x^2} e^{-1/x} dx$$

$\int x^{-2}$

$\frac{x^{-2+1}}{-2+1}$

$= \frac{-1}{x}$

$\int u\,dv = uv - u'v_1 + u''v_2 \ldots$

Let $\frac{1}{x} = u$

$$\boxed{-\frac{1}{x^2} dx = du}$$

$\frac{x^{-1-1}}{-1-0}$

$\frac{1}{x^2}$

$$\therefore \quad v = -\int e^{u} du$$

$$= -e^{u}$$

$$= -e^{1/x} \quad \left[\because u = 1/x\right]$$

$\therefore$ Second independent solution

$y_2$ is given by. $y_2 = v y_1 = x$

$$y_2 = -x \cdot e^{1/x}$$

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Solve the Euler's equation

$x^2y'' + pxy' + qy = 0.$

Soln.

Given $x^2y'' + pxy' + qy = 0 \longrightarrow (1).$

$$y'' + \frac{pxy'}{x^2} + \frac{qy}{x^2} = 0$$

$$y'' + \frac{P}{x^2} y' + \frac{q}{x^2} y = 0.$$

$P(x) = P/x^2, \quad Q(x) = q/x^2$

$P(x), Q(x)$ are not analytic at $x = 0.$

Hence $x = 0$ is a singular point

$x P(x) = P, \quad x^2 Q(x) = q,$

Which are analytic at $x = 0.$

$\therefore x = 0$ is a regular singular point.

put $z = \log x$ (or) $x = e^z$

$x = e^{\log x}$ $x = e^z$

the $\dfrac{dz}{dx} = \dfrac{1}{x}$

$$y' = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \frac{1}{x}$$

$$\therefore x\,dx = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

uv formula

$$= \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{1}{x}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{dz} + \left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{d}{dz}\left(\frac{dy}{dz}\right)\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \longrightarrow (2).$$

$\therefore$ (1) becomes,

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + p\frac{dy}{dz} + qy = 0$$

$$\frac{d^2y}{dz^2} - (p-1)\frac{dy}{dz} + qy = 0 \longrightarrow (3)$$

The auxiliary equation is

$$m^2 + (p-1)m + q = 0 \longrightarrow (4)$$

Let $m_1$ and $m_2$ be two solutions of (4). Then $e^{m_1 z}$ and $e^{m_2 z}$ are two linearly independent solutions of (3) if $m_1 \neq m_2$.

(i.e) $x^{m_1}$ and $x^{m_2}$ are two linearly independent solutions of (1) if $m_1 \neq m_2$ $\{\because x = e^z\}$

$x^{m_1}$ and $x^{m_1}\log x$ are two independent solutions if $m_1 = m_2$.

$y = Ae^{m_1x} + Be^{m_2x}$

$y = e^{mx}(Ax+B)$

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Remark.

1.) The most general solution of differential equation with regular singular point at the origin is $y'' + P/x\, y' + q/x^2\, y = 0$ with power series

$$y'' + \frac{(p_0 + p_1 x + \dots)}{x} y' + \frac{(q_0 + q_1 x + \dots)}{x^2} y = 0$$

2.) Consider the equation

$$y'' + P(x) y' + Q(x) y = 0 \longrightarrow (1)$$

The general form of the function analytic at $x=0$ is

$a_0 + a_1 x + a_2 x^2 + \cdots$

The series soln of (1) of the form

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$$

is called frobenius series.

---

Solve the equation $2x^2 y'' + x(2x+1) y' - y = 0$ by using the Frobenius method.

Soln:

Given $2x^2 y'' + x(2x+1) y' - y = 0 \longrightarrow (1)$

$$y'' + \frac{x(2x+1)}{2x^2} y' - \frac{y}{2x^2} = 0.$$

$$P(x) = \frac{2x+1}{2x}, \quad Q(x) = \frac{-1}{2x^2}$$

$P(x)$, $Q(x)$ are not analytic at $x=0$.

$x p(x) = \frac{2x+1}{2}$ , $x^2 Q(x) = \frac{-1}{2}$

which are analytic at $x=0$

$\therefore x=0$ is a regular singular point

Assume $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots$

$a_0 \neq 0$. be the Frobenius series of ①.

$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \dots$

$y'' = m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \dots$

① $\Rightarrow$ $y'' + \frac{2x+1}{2x} y' - \frac{1}{2x^2} y = 0$.

$\frac{2x+1}{2x}$

$\frac{2x}{2x} + \frac{1}{2x}$

$1 + 1/2x$

$m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \dots$

$+ (1 + 1/2x)\left(m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \dots\right)$

$- \frac{1}{2x^2}\left(a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots\right) = 0$

$m(m-1)\, a_0 x^{m-2} + m(m+1)\, a_1 x^{m-1}$

$+ (m+1)(m+2)\, a_2 x^{m} + \cdots$

$+ \left(\frac{2x+1}{2}\right)\left(m a_0 x^{m-2} + (m+1)\, a_1 x^{m-1}\right.$

$\left.(m+2)\, a_2 x^{m} + \cdots\right)$

$- \frac{1}{2}\left(a_0 x^{m-2} + a_1 x^{m-1} + a_2 x^{m} + \cdots\right) = 0$

Divide by $x^{m-2}$

$[m(m-1)\, a_0 + (m)(m+1)\, a_1 x$

$+ (m+1)(m+2)\, a_2 x^2 + \cdots]$

$+ (x + 1/2)\, [m a_0 + (m+1)\, a_1 x$

$+ (m+2)\, a_2 x^2 + \cdots]$

$- \frac{1}{2}\, [a_0 + a_1 x + a_2 x^2 + \cdots] = 0$

Equate the coefficients and Constant of $x, x^2, x^3, \ldots$ to zero

indicial equation

$m(m-1)\, a_0 + \frac{m}{2} a_0 - \frac{1}{2} a_0 = 0 \longrightarrow (2)$

$m(m+1)\, a_1 + m a_0 + \frac{m+1}{2} a_1 - \frac{a_1}{2} = 0 \longrightarrow (3)$

$(2) \Rightarrow \quad a_0\left[m(m-1) + \frac{m}{2} - \frac{1}{2}\right] = 0.$

$\Rightarrow \quad m(m-1) + \frac{m}{2} - \frac{1}{2} = 0 \quad \{\because a_0 \neq 0\}$

$$m^2 - m + \frac{m}{2} - \frac{1}{2} = 0$$

~~2m~~6.

$$2m^2 - 2m + m - 1 = 0$$

$$2m(m-1) + (m-1) = 0$$

$$(2m+1)(m-1) = 0$$

$$m = 1, -1/2$$

put $m=1$ and assume that

$a_0 = 1$.

③ $\Rightarrow 1(1+1)a_1 + \frac{1+2}{2}a_1 - \frac{a_1}{2} = 0$

$$2a_1 + 1 + a_1 - \frac{a_1}{2} = 0$$

$$3a_1 - \frac{a_1}{2} = -1$$

$$\frac{6a_1 - a_1}{2} = -1$$

$$5a_1 = -2$$

$$a_1 = -2/5$$

④ $\Rightarrow 1(1+1)(1+2)a_2 + (1+1)(-2/5)$

$$+ \left(\frac{1+2}{2}\right)a_2 - \frac{a_2}{2} = 0.$$

$$14a_2 = 8/5$$

$$a_2 = 4/35$$

Take $m = -1/2$ and assume $a_0 = 1$

(3) $\Rightarrow$ $\frac{-1}{2}\left(\frac{-1}{2}+1\right) a_1 + \left(\frac{-1}{2}\right)(1)$

$$+ \frac{(-1/2+1)}{2} a_1 - \frac{a_1}{2} = 0.$$

$a_1 = -1$

(4) $\Rightarrow$ $\left(\frac{-1}{2}+1\right)\left(\frac{-1}{2}+2\right) a_2 + \left(\frac{-1}{2}+1\right)(-1)$

$$+ \frac{(-1/2+2)}{2} a_2 - \frac{1}{2} a_2 = 0$$

$a_2 = 1/2$

Take two frobenius series solution are

$\left(x^m - \frac{2}{5} x^{m+1} + \frac{4}{35} x^{m+2} + \cdots\right)$ and

$\left(x^m - x^{m+1} + 1/2\, x^{m+2} + \cdots\right)$

Clearly the two solutions are linearly independent.

$\therefore$ The general solution of (1)

$$y = c_1\left(x^m - 2/5\, x^{m+1} + \frac{4}{35} x^{m+2} + \cdots\right)$$

$$+ c_2\left(x^m - x^{m+1} + 1/2\, x^{m+2} + \cdots\right)$$

where $c_1$ and $c_2$ are constant

Problem

$$2xy'' + (3-x)y' - y = 0$$

Soln,

Given, $2xy'' + (3-x)y' - y = 0$

$$y'' + \frac{(3-x)}{2x}y' - \frac{y}{2x} = 0$$

$$P(x) = \frac{3-x}{2x}, \quad Q(x) = \frac{1}{2x}$$

$P(x)$ and $Q(x)$ are not analytic at $x=0$

Hence $x=0$ is a singular point

$(x-x_0)P(x) = x,$

$$P(x) = x \cdot \frac{3-x}{2x} = \frac{3-x}{2}$$

$(x-x_0)^2 Q(x) = x^2,$

$$Q(x) = x^2 \cdot \frac{1}{2x} = x/2$$

$xP(x)$ and $x^2 Q(x)$ are analytic at $x=0$

Hence $x=0$ is regular singular point.

Assume,

$$y = a_0 x^m + a_1 x^{m+1} + \dots + a_0 \neq 0.$$

be the frobenius series solution for ①,

$$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{(m+1)}$$

$$y'' = m(m-1) a_0 x^{m-2} + (m)(m+1) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \dots$$

$$① \Rightarrow \quad y'' + \left(\frac{3-x}{2x}\right) y' - \frac{1}{2x} y = 0.$$

$$m(m-1) a_0 x^{m-2} + m(m-1) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \dots + \left(\frac{3}{2x} - \frac{x}{2x}\right) m a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots \Big] = 0$$

$$\Big[ m(m-1) a_0 x^{m-2} + m(m-1) a_1 x^{m-1} + (m+1)(m+2) a_2 x^m + \dots - \tfrac{1}{2} m a_0 x^{m-1} - \tfrac{1}{2}(m+1) a_1 x^m - \tfrac{1}{2} (m+1)(a_2) x^{m+1} + \tfrac{1}{2} a_0 x^{m-1} + \tfrac{1}{2} a_1 x^m + \tfrac{1}{2} a_2 x^{m+1} + \dots \Big] = 0$$

$m(m-1)a_0x^{m-2} + m(m+1)a_1x^{m-1}$

$+(m+1)(m+2)a_2x^m + \dots + 3/2\, m a_0 x^{m-2}$

$+ 3/2\, x^{m+1} a_1 x^{m-1} + 3/2\,(m+2) a_2 x^m + \dots$

$- 1/2\, m a_0 x^{m-1} - 1/2\,(m+1) a_1 x^m -$

$1/2\,(m+1) a_2 x^{m+1} - 1/2\, a_0 x^{m-1}$

$- 1/2\, a_1 x^m - 1/2\, a_2 x^{m+1} = 0.$

Divide $x^{m-2}$,

$m(m-1)a_0 + m(m+1)x + (m+1)(m+2)a_2x^2$

$+ 3/2\, m a_0 + 3/2\,(m+1) a_1 x^2 - 1/2\,(m+1) a_2 x^3$

$-1/2\, a_0 x - 1/2\, a_1 x^2 - 1/2\, x^3 = 0.$

Equating the coefficients and constant of $x, x^2 \dots$ to zero

$m(m-1)\, a_0 + 3/2\, m a_0 = 0.$ —— (1)

$m(m+1) + 3/2\,(m+1) a_1 - 1/2\, m a_0$

$- 1/2\, a_0 = 0$ —— (2)

$(m+1)(m+2)a_0 + 3/2\,(m+2) a_2 - 1/2\,(m+1) a_1$

$- 1/2\, a_0 = 0$ —— (3)

① $\Rightarrow$ $a_0\left[m(m-1)+3/2\,m\right]=0.$

$m^2-m+3/2\,m=0$

$2m^2-2m+3m=0$

$2m^2+m=0 \Rightarrow m(2m+1)=0$

$\Rightarrow 2m+1=0$

$m=0,\ -1/2$

put $m=0$ and assume $a_0=1$

② $\Rightarrow$ $0(0+1)+3/2\ (0+1)\ a_1-1/2(0)(1)-1/2(1)=0$

$0+3/2\ a_1-1/2=0$

$1-3/2\,a_1=0$

$a_1=1/2\times 2/3 \Rightarrow a_1=1/3$

put $m=0,$

③ $\Rightarrow$ $2a_2+\frac{3}{2}(2)\,a_2-1/2(1)(1/3)-1/2\cdot 1/3=0$

$2a_2+3a_2-1/6-1/6=0$

$5a_2-2/6=0$

$5a_2-1/3=0$

$a_2=1/3\times 1/5=1/15$

$\Rightarrow a_2=1/15$

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$m=0, \quad a_0 = 1, \quad a_1 = 1/3, \quad a_2 = 1/15$

$\Rightarrow m = -1/2$

$\Rightarrow \frac{-1}{2}\left(-\frac{1}{2}+1\right) + 3/2\left(-1/2+1\right)a_1 - 1/2(-1/2)\, a_0$

$- 1/2\, a_0 = 0$

$-1/2\,(1/2) + 3/2\,(1/2)\, a_1 + 1/4\, a_0 - 1/2\, a_0 = 0$

$-\frac{1}{4} + 3/4\, a_1 + 1/4\, a_0 - 1/2\, a_0 = 0.$

$3/4\, a_1 - 1/4\, a_0 = 1/4$

$3/4\, a_1 = 1/4 + 1/4$

$3/4\, a_1 = 2/4$

$\Rightarrow 3a_1 = 2$

$\Rightarrow \boxed{a_1 = 3/3}$

$m = -1/2$

$\Rightarrow \left(-1/2+1\right)\left(-\frac{1}{2}+2\right)a_2 + 3/2\left(-1/2+2\right)a_2$

$-\frac{1}{2}\left(-1/2+1\right) - 1/2\, a_1 = 0$

$\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)a_2 + (3/2)(3/2)\, a_2 - 1/2\,(1/2)(2/3)$

$-1/2\,(2/3) = 0$

$3/4 \, a_2 + a/4 - 1/6 - 1/3 = 0$

$$\frac{12}{4} a_2 - 3/6 = 0$$

$$\frac{12}{4} a_2 = 1/2$$

$$\boxed{a_2 = 1/6}$$

$m = -1/2, \quad a_0 = 1, \quad a_1 = 2/3, \quad a_2 = 1/6$

---

Problem:

$x^2 y'' - 3xy' + (4x+4)y = 0$. Show that has only one frobenius series and find it.

Soln:

Given,

$$x^2 y'' - 3xy' + (4x+4)y = 0$$

$$\Rightarrow y'' - \frac{3xy'}{x^2} + \frac{(4x+4)}{x^2} y = 0$$

$$\Rightarrow y'' - \frac{3}{x} y' + \frac{(4x+4)}{x^2} y = 0$$

$P(x) = -3/x$ and $Q(x) = \frac{4x+4}{x^2}$

$P(x)$ and $Q(x)$ are analytic

at $x=0$.

Here $x=0$ is singular point

$$xp(x) = x\left(-3/x\right) = -3$$

$$x^2 Q(x) = \frac{x^2(4x+4)}{x^2} = 4x+4$$

$xp(x)$ and $x^2 Q(x)$ are analytic

at $x=0$.

$x=0$ is regular singular point.

Assume that

$$y = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \dots$$

be a frobenius series soln for eqn ①

$$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1}$$

$$y'' = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+1)(m+2) a_2 x^m$$

$\cancel{x^2 y'' \rightarrow}$

$$y'' - \frac{3y''}{x} + \frac{4x+4}{x^2} y = 0$$

$$\Big[ m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1}$$

$$+ (m+1)(m+2) a_2 x^m + \cancel{3x} \cdots$$

$$- 3/x \left[ m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \cdots \right]$$

$$+ \frac{4x+4}{x^2} \left[ a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots \right] = 0$$

$$m(m-1) a_0 x^{m-2} + m(m+1) a_1 x^{m-1}$$

$$+ (m+1)(m+2) a_2 x^m + \cdots - 3m a_0 x^{m-2}$$

$$- 3(m+1) a_1 x^{m-1} - 3(m+2) a_2 x^m - \cdots$$

$$4a_0 x^{m-1} + 4a_1 x^m + 4a_2 x^{m+1} + \cdots + 4a_0 x^{m-2}$$

$$+ 4a_0 x^m + 4a_2 x^m + \cdots = 0$$

Divide $x^{m-2}$

$$m(m-1) a_0 + m(m+1) a_1 x + (m+1)(m+2) a_2 x^2$$

$$- 3m a_0 - 3(m+1) a_1 x - 3(m+2) a_2 x^2 + \cdots$$

$$- 4a_0 x + 4a_1 x^2 + 4a_2 x^3 + \cdots + 4a_0 \cancel{x}$$

$$+ 4a_1 x + 4a_2 x^2 + \cdots = 0.$$

Equating the coefficients of

$x, x^2, \ldots$ to zero

$m(m-1)\, a_0 - 3m a_0 + 4a_0 = 0 \longrightarrow (2)$

$m(m+1)\, a_1 - 3(m+1)\, a_1 + 4a_0 + 4a_1 = 0 \longrightarrow (3)$

$(m+1)(m+2)\, a_2 - 3(m+2)\, a_2 + 4a_1 + 4a_2 \; (4)$

$(2) \Rightarrow a_0 \left[ m(m-1) - 3m + 4 \right] = 0$

$m^2 - m - 3m + 4 = 0$

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0 \qquad \Rightarrow m - 2 = 0$

$m = 2$

$m = 2, 0$

4

4

$-2 \; -2$

## Legendre Polynomial

Derive the $n^{th}$ degree polynomial $P_n(x)$ and deduce that $P_n(1) = 1$, $P_n(-1) = (-1)^n$.

Soln:-

Consider the Legendre polynomial

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \rightarrow (1)$$

The equation is analytic in the region $-1 < x < 1$.

To find the soln of (1) bounded near $x = 1$.

now, let $t = \frac{1}{2}(1-x) \Rightarrow \frac{1}{2} - \frac{1}{2}x$

$2t = 1-x$ $\qquad dt = -\frac{1}{2}dx$

$x = 1-2t$

$x = -2t+1 \Rightarrow \frac{dt}{dx} = -\frac{1}{2}$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{1}{2}\frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{1}{2}\frac{dy}{dt}\right)$$

$$= -\frac{1}{2} \frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= -\frac{1}{2} \frac{d}{dt}\left(-\frac{1}{2}\frac{dy}{dt}\right)$$

$$= \frac{1}{4}\left(\frac{d^2y}{dt^2}\right)$$

From ①, we get

$$\left(1-(1-2t)^2\right)\left(\frac{1}{4}\frac{d^2y}{dt^2}\right) - 2(1-2t)\left(\frac{-1}{2}\cdot\frac{dy}{dt}\right)$$

$1-(1^2-4t+4t^2)$

$$+ n(n+1)y = 0$$

$1-1+4t-4t^2$

$$(4t-4t^2)\left(\frac{1}{4}\frac{d^2y}{dt^2}\right) + (1-2t)\frac{dy}{dt}$$

$$+ n(n+1)y = 0$$

$$\cancel{4}t(1-t)\frac{1}{\cancel{4}}\left(\frac{d^2y}{dt^2}\right) + (1-2t)\frac{dy}{dt} + n(n+1)y = 0$$

$$t(1-t)\frac{d^2y}{dt^2} + (1-2t)\frac{dy}{dt} + n(n+1)y = 0$$

where $y' = \frac{dy}{dt}$ , $y'' = \frac{d^2y}{dt^2} \rightarrow ②$

This is the Hyper Geometric equation with $a=n$, $b=n+1$, $c=1$ and near to $t=0$

The solution of equation (2) is

$$y_1 = F(\overset{a}{-n}, \overset{b}{n+1}, \overset{c}{1}, \overset{d}{t})$$

The second soln of (2) in $y_2 = vy$,

where, $v = \int \frac{1}{y_1^2} e^{-\int p(t)\,dt} dt$ $\quad -\int p\,dx$

$$= \int \frac{1}{y_1^2} e^{-\int \frac{1-2t}{t(1-t)} dt} \cdot dt \qquad P(t) = \frac{y'}{y''}$$

$$v' = \frac{1}{y_1^2} e^{\log(t(1-t))^{-1}}$$

$$= \frac{1}{y_1^2} \frac{1}{(-(1-t))}$$

$y_1$ is a polynomial with constant term

$$v' = 1/t\,(1 + a_1 t + a_2 t^2 + \dots)$$

$$= \frac{1}{t} + a_1 + a_2 t + a_3 t^2 + \dots$$

Integrating, we get

$$v = \log t + a_1 t + a_2 \frac{t^2}{2} + \frac{a_3 t^3}{3} + \dots$$

$\therefore$ the soln of eqn (2) is

$$y = C_1 y_1 + C_2 y_2 \longrightarrow (3)$$

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Because of the present, in $\log t$ is $y_2$ it is clear that 3 is bounded near $t=0$ if $c_2=0$.

If we replace $t$ by $\frac{1}{2}(1-x)$ the solution of (1) is bounded near $x=1$ and are constant multiply of polynomial.

The $n^{th}$ degree polynomial $P_n(x)$ is defined by

$$P_n(x) = F(-n, n+1, 1, \tfrac{1}{2}, 1-x)$$

$$= 1 + \frac{(-n)(n+1)}{1!\,(1)}\left(\frac{1-x}{2}\right) + \frac{-n(-n+1)(n+1)(n+2)}{n!\,(1^2)} + \cdots$$

$$+ \frac{n(n-1)\cdots(n-(n-1))(n+1)(n+2)}{n!\,(1,2,\ldots,n)}$$

$$= 1 + \frac{n(n+1)}{1!}\left(\frac{x-1}{2}\right) + \frac{n(n-1)(n+1)(n+2)}{(n!)^2\,2^3}(x-1)^2 + \cdots$$

$$+ \frac{n(n-1)\cdots(n-(n+1))\,(n+1)\cdots 2n}{(n!)^2\,2^n} + \cdots$$

$$P_n(x) = \frac{1+n(n+1)}{1!\,2^{(1)}}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2\,2^n}(x-1)^2 + \dots$$

$$+ \frac{n(n-1)\dots 1\,(n+1)(n+2)\dots(2n)}{(n!)^2\,2^n}(x-1)^n$$

$$P_n(x) = 1 + \frac{n(n+1)}{1!\,2!}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2\,2^2} + \dots$$

$$+ \frac{(2n)!}{(n!)^2\,2^n}(x-1)^n \longrightarrow (4)$$

$P_n(x)$ is a polynomial of degree $n$, that contains only even or odd powers $n$ accordinly $n$ is even or odd.

$\therefore$ It can be written as

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \dots \rightarrow (5)$$

This ends with $a_0$ if $n$ is even and $a_1$ if $n$ is odd

$$P_n(x) = 1 + \frac{n(n+1)}{1!\,2}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2 \cdot 2^2}(x-1)^2 + \dots$$

put $x=1$, we get

$P_n(1)=1$

put $x=-1$, we get in (1),

$P_n(-1) = a_1(-1)^n + a_{n-2}(-1)^{n-2} + a_{n-4}(-1)^{n-4} + \dots$

$= (-1)^n (a_1 + a_{n-2} + a_{n-4} + \dots)$

$= (-1)^n \quad \{\because P_n(1) = 1\}$

$P_n(-1) = (-1)^n$

//

State and Prove orthogonal property of Legendre polynomials

V.Q

Prove that

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

where the Sequence of Legendre polynomial $P_0(x), P_1(x), P_2(x), \dots, P_n(x)$ is a sequence of orthogonal function on the interval $-1 \leq x \leq 1$

consider, $I = \int_{-1}^{1} f(x) P_n(x)\, dx$

By Rodrique's formula, we have

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

$$\therefore I = \frac{1}{2^n . n!} \int_{-1}^{1} f(x) . \frac{d^n}{dx^n} (x^2-1)^n \, dx$$

$$= \frac{1}{2^n . n!} \int_{-1}^{1} f(x) \frac{d}{dx} \left[ \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right] dx$$

$$= \frac{1}{2^n . n!} \; f(x) \left[ \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right]_{-1}^{1}$$

$$- \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n f'(x) \, dx \; \Big\}$$

$$= \frac{1}{2^n . n!} \left\{ 0 - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n f'(x) \, dx \right.$$

$$\therefore I = \frac{-1}{2^n . n!} \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n f'(x) \, dx$$

Proceeding like this, we get

$$I = \frac{(-1)^n}{2^n . n!} \int_{-1}^{1} \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n f'(x) \, dx$$

$$I = \frac{(-1)^n}{2^n . n!} \int_{-1}^{1} (x^2-1)^n f^n(x)\,dx \rightarrow (1)$$

Assume that

$$\underline{\underline{f(x) = P_m(x)}}$$

w.L.G, assume that $m < n$

Then $f^n(x) = 0 \Rightarrow f^n(x) = P_m(x) = 0$

$$\therefore I = \int_{-1}^{1} P_m(x)\, P_n(x)\,dx = 0$$

If $m = n$, put $f(x) = P_n(x)$

$$(1) \Rightarrow \quad I = \frac{(-1)^n}{2^n . n!} \int_{-1}^{1} P_n^{(n)}(x)(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^n . n!} \frac{(2n)!}{(n!)\,2^n} \int_{-1}^{1} (x^2-1)^n\,dx$$

$$\left\{ \because P_n^{n}(x) = \frac{2n!}{(n!)\,2^n} \right\}$$

$$= \frac{(-1)^n (2n!)}{2^{2n} . (n!)^2} \int_{-1}^{1} (x^2-1)^n\,dx$$

$$= \frac{(-1)^n (2n)!}{2^{2n}\cdot(n!)^2}\, 2\int_0^1 (x^2-1)^n\, dx$$

put $\cancel{x} = \sin\theta$, then $dx = \cos\theta\, d\theta$

| | | |
|---|---|---|
| $x$ | $0$ | $1$ |
| $\theta$ | $0$ | $\pi/2$ |

$$I = \frac{(-1)^n (2n)!}{2^{2n}\cdot(n!)^2}\, 2\int_{-1}^{\pi/2} (\sin^2\theta-1)^n \cos\theta\, d\theta$$

$(-1)^{2n}$ = +ve value

$$= \frac{(-1)^n (2n)!}{2^{2n}\cdot(n!)^2}\, 2\int_0^{\pi/2} (-1)^n \cos^{2n}\theta \cos\theta\, d\theta$$

$$= \frac{(2n)!}{2^{2n}\cdot(n!)^2}\, 2\int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

$$\boxed{\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \sin^2\theta - 1 &= \cos^2\theta\end{aligned}}$$

$$\left[\therefore \int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\cdot 1 \text{ if } n \text{ is odd}\right]$$

$$= \frac{(2n)!}{2^{2n}(n!)}\, 2\left[\frac{2n}{(2n-1)}\,\frac{(2n-2)}{(2n-3)}\cdots\frac{2}{3}\right]$$

$$= \frac{(2n)!}{2^{2n}\cdot n!}\, 2\, \frac{(2)^{2n}\,(n!)^2}{(2n+1)\,(2n)!}$$

$$= \frac{2}{2n+1}$$

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

Derive Rodrigue's formula for Legendre polynomial

<u>proof:-</u>

W.k.T the Recurrsion formula for Legendre polynomial

$$a_{n+2} = \frac{-(p-n)(n+p+1)}{(n+1)(n+2)} a_n.$$

Replace $p$ by $n$ and $n$ by $k-2$

$$a_k = \frac{-(n-k+2)(k-2+n+1)}{(k-2+1)(k-2+2)} a_{k-2}$$

$$= \frac{-(n-k+2)(n+k-1)}{(k-1)(k)} a_{k-2}$$

$$a_{k-2} = \frac{-k(k-1)}{(n+2-k)(k+n-1)} a_k$$

$$= \frac{-k(k-1)}{(n-k+2)(k+n-1)} a_k .$$

when $k=n, n-2, n-4, \ldots$

we have

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

$$a_{n-4} = \frac{(n-2)(n-2-1)}{(n-n+2+2)(n-2+n-1)} a_{n-2}$$

$$= \frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$a_{n-4} = \frac{(n-2)(n-3)}{4(2n-3)} \cdot \frac{-n(n-1)}{2(2n-1)} a_n$$

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$$= \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} a_n \text{, and so on.}$$

Now,

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \dots$$

$$= a_n x^n + \frac{n(n-1)}{2(2n-1)} x^{n-2} a_n$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x a_n$$

$$= a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x a_n + \dots$$

$$+ \frac{(-1)^k n(n+1) \cdots (n-(2k-1)) a_n x^{n-2k}}{(2 \cdot 4 \cdots 2k)(2n-1) \cdots (2n-(2k-1))}.$$

$$= a_n \left[ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdots (2n-1)(2n-3)} x^{n-4} + \dots + \frac{(-1)^k n(n-1) \cdots (n-2k+1)}{2 \cdot 4 \cdots \cancel{(2n-2k+1)}\, 2k-1} \right.$$

$$2k(2n-1)\cdots(2n-2k+1)\,n(n-1)\cdots(n-2k+1)\,x^{n-2k}+\cdots$$

$\text{— (1)}$

Now $n(n-1)\cdots(n-2k+1)$

$$=\frac{n(n-1)\cdots(n-2k+1)\,(1\cdot 2\cdots n-2k)}{(1\cdot 2\cdots\cdots n-2k)!}$$

$$=\frac{n(n-1)\cdots\cdots(n-2k+1)(n-2k)}{(n-2k)!}$$

$$=\frac{n!}{(n-2k)!}$$

$$2\cdot 4\cdot 6\cdots\cdots 2^k = 2^k\,(1\cdot 2\cdots k)$$

$$=2^k\,(k!)$$

$$2^n\,(2n-1)(2n-3)\cdots\cdots(2n-2k+1)$$

$$=\frac{2^n(2n-1)(2n-2)\cdots\cdots(2n-2k+2)(2n-2k+1)}{2n\cdots(2n-2)(2n-4)\cdots(2n-2k+2)}$$

$$=\frac{(2n)!}{(1\cdot 2\cdots 2n-2k)\,(2n)\,(2n-2)(2n-4)\cdots(2n-2k+2)}$$

$$= \frac{2n!}{(2n-2n)! \; \frac{(n)(n-1)(n-2)\cdots(n-k+1)(n-k)!}{(n-k)!}}$$

$$= \frac{2n!}{(2n-2k)! \; \frac{2^k (n!)}{(n-k)!}}$$

$$P_n(x) = a_n x^n - \frac{n(n-1)}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{n-4} + \dots + \frac{(-1)^k \frac{n!}{(n-2k)!}}{2^k (k!) \frac{2n!}{(2n-2k)!} \frac{2^k (n!)}{(n-k)!}} x^{n-2k} + \dots$$

The coefficient of $x^{n-2k}$

$$= \frac{(-1)^K \frac{n!}{(n-2k)!}}{2^k (k!) \frac{2n!}{(2n-2k)! \; \frac{2^k n!}{(n-k)!}}}$$

$$= \frac{(-1)^k n! \, (2n-2k)! \; 2^k (n!)}{(n-2k)! \, 2^k (k!) \, 2n! \, (n-k)!}$$

$$= \frac{(-1)^k (n!)^2 (2n-2k)!}{2n! \, k! \, (n-k)! \, (n-2k)!}$$

The eqn (H), $a_n = \frac{2n!}{(n!)^2 \, 2^n}$.

value $[n/2]$ is the usual symbol for the greatest $\leq n/2$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{2^n . k! \, (n-k) n!} \frac{d^n}{dx^n} (x^{2n-2k})$$

$$\frac{d^n}{dx^n} (x^{2n-2k}) = \frac{d^{n-1}}{dx^{n-1}} d/dx \, (x^{2n-2k})$$

$$= \frac{d^{n-1}}{dx^{n-1}} (2n-2k) \, x^{2n-2k-1}$$

$$= \frac{d^{n-2}}{dx^{n-2}} (2n-2k-1)(2n-2k) \, x^{2n-2k-2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= \frac{d^{n(n-1)}}{dx^{n(n-1)}} (2n-2k)(2n-2k-1) \ldots (2n-2k-(n-2)) \, (2n-2k) \ldots (n-1)x$$

$$= \frac{d}{dx} (2n-2k)(2n-2k-1) \cdots (n-2k+2)(x^{n-2k+1})$$

$$= \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

$$= \frac{1}{2^n . n!} \frac{d^n}{dx^n} \sum_{k=0}^{n} \frac{(-1)^k . n!}{k!(n-k)!} x^{2n-2k}$$

[$\because$ terms less than $n$ are 0 for $n^{th}$ derivations]

$$= \frac{1}{2^n . n!} \frac{d^n}{dx^n} \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} (x^2)^{n-k}$$

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n$$

which is called the Rodrigues formula for legendre equation

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = 1/2 (3x^2 - 1)$

$P_3(x) = 1/2 (5x^3 - 3x)$ //.

Problem

The function on the left side of $\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x)t +$ $P_2(x^2)t^2 + P_3(x)t^3 + \dots + P_n(x)t^n + \dots$ is called the generating function of the legendre polynomial. Assume that this relation is three and use it ~~relation is~~

(a) To verify that $P_n(1) = 1$, $P_n(-1) = (-1)^n$

(b) S.t $P_{2n+1}(0) = 0$ and

$$P_{2n}(0) = \frac{(-1)^n \, 1.3.\dots(2n-1)}{2^n \cdot n!}$$

Proof.

$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x)t + \dots \longrightarrow (1)$$

Put $x = 1$, we get

$$\frac{1}{\sqrt{1-2t+t^2}} = P_0(1) + P_1(1)t + P_2(1)t^2 + \dots + P_n(1)t^n + \dots$$

$$\frac{1}{\sqrt{(1-t)^2}} = P_0(1) + P_1(1)t + P_2(1)t^2 + \dots P_n(1)t^n + \dots$$

$t^2-2t+1$

$$(1-t)^{-1} = P_0(1) + P_1(1)t + P_2(1)t^2 + \dots + P_n(1)t^n + \dots \cancel{+ P_n(1)t^n} + \dots$$

$(1-t)$
$(1-t)$
$(1-t)^2$

$$1+t+t^2+\dots = P_0(1) + P_1(1)t + P_2(1)t^2 + \dots + (P_n(1)t^n + \dots)$$

Equating the coefficients term of $t^n$.

$P_n(1) = 1.$

Put $x = -1$

$$\frac{1}{\sqrt{1+2t+t^2}} = P_0(-1) + P_1(-1)t + P_2(-1)t^2 + \dots + P_n(-1)t^n + \dots$$

$$\frac{1}{\sqrt{(1+t)^2}} = P_0(-1) + P_1(-1)t + P_2(-1)t^2 + \dots + P_n(-1)t^n + \dots$$

$$(1+t)^{-1} = P_0(-1) + P_1(-1)t + P_2(-1)t^2 + \dots + P_n(-1)t^n + \dots$$

$$1-t+t^2-t^3+\dots = P_0(-1) + P_1(-1)t + P_2(-1)t^2 + \dots + P_n(-1)t^n + \dots$$

Equating the coefficients term

of $P_n(-1) = (-1)^n$

(b) put $x=0$ in ①

$$\frac{1}{\sqrt{(1+t)^2}} = P_0(0) + P_1(0)t + P_2(0)t^2 + \dots + P_n(0)t^n + \dots$$

$$\frac{1}{(1+t^2)^{1/2}} = P_0(0) + P_1(0)t + P_2(0)t^2 + \dots + P_n(0)t^n + \dots$$

$$(1+t^2)^{-1/2} = P_0(0) + P_1(0) + \dots + P_n(0)t^n + \dots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$P_0(0) + P_1(0)t + P_2(0)t^2 + \dots + P_n(0)t^n + \dots$$
$$= 1 - (1/2)t^2 + \frac{(-1/2)(-1/2-1)}{2!}t^4 - \frac{(-1/2)(-1/2-1)(-3/2)}{3!}t^6$$

The expression on right side contains only even power of t.

$P_{2n+1}(0) = 0$

Equating coefficient of $t^{2n}$,

we get,

$$P_{2n}(0) = \frac{(-1/2)(-1/2-1)(-1/2-2)\cdots(-1/2-(2n-1))}{n!}$$

$$= \frac{(-1)^n (1 \cdot 3 \cdot 5 \cdots 2n-1)}{2^n \cdot n!}$$

11/10/19

Consider the generating relation

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

(a) By differentiating both sides with respect to 't' show that

$$(x-t)\sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} n P_n(x)\, t^{n-1}$$

(b) Equating the coefficients of $t^n$ in (a) ~~both~~ obtain

$$(n+1)\, P_{n+1}(x) = (2n+1)\, x P_n(x) - n(P_{n-1}(x))$$

(c) Assume that $P_0(x) = 1$, $P_1(x) = x$ are known and write the recurssion formula in (b) to calculate $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$

Soln

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \rightarrow (1)$$

Diff. w.r. to 't' on both sides, we have

$$-\frac{1}{2}(-2xt+t^2)^{-3/2}\cdot(-2x+2t)=\sum_{n=1}^{\infty} nP_n(x)\,t^{n-1}$$

$$\left(-\frac{1}{2}\right)\frac{1}{(1-2xt+t^2)^{3/2}}(-2x+2t)=\sum_{n=1}^{\infty} nP_n(x)\,t^{n-1}$$

$$\left(-\frac{1}{2}\right)\frac{1}{(1-2xt+t^2)^{3/2}}(-2)(x-t)=\sum_{n=1}^{\infty} nP_n(x)\,t^{n-1}$$

$$\frac{x-t}{\sqrt{(1-2x+t^2)(1-2xt+t^2)2}}=\sum_{n=1}^{\infty} nP_n(x)\,t^{n-1}$$

$$(x-t)\frac{1}{\sqrt{1-2xt+t^2}}=(1+2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)\,t^{n-1}$$

$$(x-t)\sum_{n=0}^{\infty} P_n(x)^n=1+2xt+t^2\sum_{n=1}^{\infty} nP_n(x)\,t^{n-1}\quad[\text{by } ①]$$

(b) coefficient of $t^n$ in L.H.S

$=$ coefficient of $t^n$ in $(x-t)\left[P_0(x)+P_1(x)t+P_2(x)t^2+\dots+P_{n-1}(x)t^{n-1}+P_n(x)t^n+\dots\right]$

$=$ coefficient of $t^n$ in $(x-t)\sum_{n=0}^{\infty} P_n(x)\,t^n$

$= xP_n(x)-P_{n-1}(x)$

coefficient of $t^n$ in R.H.S.

$=$ coefficient of $t^n$ is $1-2xt+t^2\sum_{n=1}^{\infty} nP_n(x)t^{n-1}$

$= \text{coefficient of } t^n \text{ is } (1-2xt+t^2)$

$$[P_1(x)+2P_2(x)t+3P_3(x)t^2+\cdots+$$
$$(n-1)P_{n-1}(x)t^{n-2}+\cdots+nP_n(x)t^{n-1}$$
$$+(n+1)P_{n+1}(x)t^n].$$

$$=(n+1)P_{n+1}x - \underset{\text{doub.}}{2xn}\, P_n(x)+(n-1)P_{n-1}(x)$$

Equating the coefficients.

$$xP_n(x)-P_{n-1}(x)=(n+1)P_{n+1}(x)-2x^n P_n(x)$$
$$+(n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x)=xP_n(x)-P_{n-1}(x)+2xnP_n(x)$$
$$-(n-1)P_{n-1}(x)$$

$$=xP_n(x)(1+2x)-P_{n-1}(x)(x+n-x)$$

$$=xP_n(x)(2n+1)-P_{n-1}(x)(n)$$

$$\therefore (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \rightarrow (1)$$

c) Given $P_0(x)=1$, $P_1(x)=x$.

$n=1$,

$$(1) \Rightarrow 2P_2(x)=3xP_1(x)-P_0(x)$$
$$=3x\,x-P_1$$
$$=3x^2-1$$
$$P_2(x)=\frac{3x^2-1}{2}$$

$n=2$,

$$3P_3(x) = 5x\,P_2(x) - 2P_1(x)$$

$$= 5x\left(\frac{3x^2-1}{2}\right) - 2(x)$$

$$= \frac{15x^3-5x}{2} - 2x$$

$$= \frac{15x^3-5x-4x}{2}$$

$$3P_3(x) = \frac{15x^3-9x}{2}$$

$$P_3(x) = \frac{15x^3-9x}{6}$$

$n=3$,

$$P_4(x) = 7x\,P_3(x) - 3P_2(x)$$

$$= 7x\left(\frac{15x^3-9x}{6}\right) - 3\left(\frac{3x^2-1}{2}\right)$$

$$= \frac{105x^4-63x^2}{6} - \frac{9x^2-3}{2}$$

$$= \frac{105x^4-63x^2-27x^2+9}{6}$$

$$= \frac{105x^4-90x^2+9}{6}$$

$$= \frac{\cancel{3}^{1}(35x^4-30x^2+3)}{\cancel{6}\,2}$$

$$4P_4(x) = \frac{35x^4-30x^2+3}{2}$$

$$P_4(x) = \frac{35x^4-30x^2+3}{8}$$

$n=4,$

$$5P_5(x) = 9x P_4(x) - 4P_3(x)$$

$$= \frac{9x(35x^4-30x^2+3)}{8} - \frac{\cancel{4}^{2}(15x^3-9x)}{\cancel{6}\,3}$$

$$= \frac{315x^5-270x^3+27x}{8} - \frac{30x^3-18x}{3}$$

$$= \frac{315x^5-270x^3+27x}{8} - \frac{\cancel{8}(10x^3-6x)}{\cancel{8}}$$

$$5P_5(x) = \frac{315x^5-270x^3+27x-80x^3+48x}{8}$$

$$P_5(x) = \frac{315x^5-350x^3+75x}{8\times 5}$$

$$P_5(x) = \frac{315x^5-350x^3+75x}{40}.$$

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15/10/19 - Legendre Series State and Derive!

We have the Legendre polynomial $P_0(x) = 1$, $P_1(x) = x$,

$P_2(x) = \frac{3x^2-1}{2}$, $P_3(x) = \frac{5x^3-3x}{2}$ and

so on.

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_2(x) = \frac{3x^2-1}{2}$$

$$3x^2-1 = 2P_2(x)$$

$$3x^2 = 1+2P_2(x)$$

$$\boxed{x^2 = \frac{1}{3}\left(1+2P_2(x)\right)}$$

$$\frac{5x^3-3x}{2} = P_3(x)$$

$$5x^3-3x = 2P_3(x)$$

$$5x^3 = 3x+2P_3(x)$$

$$\boxed{x^3 = \frac{1}{5}\left[3P_1(x)+2P_3(x)\right]}$$

Generally, we can write $x^n$ as linear combination of legendre polynomial.

$$p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$= b_0 (P_0(x)) + b_1 P_1(x) + b_2\left[\frac{P_0(x)}{3} + \frac{2P_2(x)}{3}\right] + b_3\left[\frac{3P_1(x)}{5} + \frac{2P_3(x)}{5}\right]$$

$$= P_0(x)\left[b_0 + \frac{b_2}{3}\right] + P_1(x)\left[b_1 + \frac{3b_3}{5}\right]$$

$$+ P_2(x)\left[\frac{2b_2}{3}\right] + P_3(x)\left[\frac{2b_3}{5}\right]$$

$$= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x)$$

$$P(x) = \sum_{n=0}^{3} a_n P_n(x)$$

In general any polynomial of degree $n$ say $P(x)$ can be written as

$$P(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots + a_n P_n(x)$$

Then $f(x)$ is arbitrary founction then the legendre polynomial is

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots + a_n P_n(x) + \cdots$$

$$\therefore f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

This is called Legendre

Series

To find $a_n$:-

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \int_{-1}^{1} \sum_{n=0}^{\infty} a_n P_n(x) P_m(x)\, dx$$

$$= \sum_{n=0}^{\infty} a_m \left(\frac{2}{2m+1}\right)$$ (by orthogonal property)

$$= a_m \left(\frac{2}{2m+1}\right)$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

$$a_n = (n+1/2) \int_{-1}^{1} f(x) P_n(x)\, dx$$ //.

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# Least square approximation

Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$, approximate $f(x)$ as closely as possible in the sense of least squares by polynomials $p(x)$ of degree $\le n$.

**proof:-**

consider,

$$I = \int_{-1}^{1} \left( f(x) - p(x) \right)^2 dx$$

which represents the sum of squares of derivations of $p(x)$ from $f(x)$.

Now to minimize the value of this integral by suitable choice of $p(x)$

For this consider the minimizing polynomial which is the

Sum of first $(n+1)$ terms of legendre Series.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x),$$

$$P(x) = \sum_{n=0}^{n} a_k P_k(x)$$

$$\therefore \; P(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots + a_n P_n(x)$$

where $a_n = n + 1/2 \int_{-1}^{1} f(x)\, P_n(x)\, dx$

Now all the polynomial of degree $< n$ are expressible in the form

$$b_0 P_0(x) + b_1 P_1(x) + \cdots + b_n P_n(x)$$

$$I = \int_{-1}^{1} (f(x) - P(x))^2 \, dx$$

$$= \int_{-1}^{1} \left( f(x) - \sum_{n=0}^{n} b_k P_k(x) \right)^2 dx$$

$$= \int_{-1}^{1} f(x)^2 . dx + \int_{-1}^{1} \left( \sum_{n=0}^{n} b_k P_k(x) \right)^2 dx$$

$$- 2 \int_{-1}^{1} f(x) \sum_{k=0}^{n} b_k P_k(x)\, dx$$

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(2) $\times P_m \Rightarrow \frac{d}{dx}\left[(1-x^2)P_n'\right]P_m + n(n+1)P_mP_n$ — (4)

(3) − (4)

$$\left[\frac{d}{dx}(1-x^2)\right]\left[P_nP_m' - P_mP_n'\right] + P_mP_n\left[m(m+1) - n(n+1)\right] = 0$$

$$\Rightarrow \left[\frac{d}{dx}(1-x^2)\right]\left[P_nP_m' - P_mP_n'\right] = \left[n(n+1) - m(m+1)\right]P_mP_n$$

$$\Rightarrow \int_{-1}^{1}\left[\frac{d}{dx}(1-x^2)\right]\left(P_nP_m' - P_mP_n'\right)dx = \int_{-1}^{1}\left[n(n+1) - m(m+1)\right]P_mP_n\,dx$$

$$\Rightarrow \left(P_nP_m' - P_mP_n'\right)\int_{-1}^{1}\frac{d}{dx}(1-x^2)\,dx = \left[n(n+1) - m(m+1)\right]\int_{-1}^{1}P_mP_n\,dx$$

$$\Rightarrow 0 = \left[n(n+1) - m(m+1)\right]\int_{-1}^{1}P_mP_n\,dx$$

$$\Rightarrow \int_{-1}^{1}P_m(x)\,P_n(x) = 0, \quad n \neq m$$

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$$\Rightarrow \int_{-1}^{1} P_m(x)\, P_n(x) = 0, \quad n \neq m$$

Problem : (3)

If the generating series

$\dfrac{1}{\sqrt{1-2xt+t^2}}$ is squared and integrated

from $x=-1$ to $x=1$ then the

first part of orthogonal property

implies that,

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}$$

Establish the Second part of the

orthogonal Property by Showing

that the integral on the - left

has the value $\sum_{n=0}^{\infty} \frac{2}{2n+1} \cdot t^{2n}$.

Soln:- Consider,

$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x) + P_2(x)t^2 + \cdots + P_n(x)t^n + \cdots$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \left[P_0(x) + P_1(x)t + P_2(x)t^2 + \dots + P_n(x)t^n + \dots\right]^2$$

$$= \left[P_0^2(x) + P_1^2(x)t^2 + \dots + P_n^2(x)t^{2n} + \dots\right]$$

$$+ 2\left[P_0(x)P_1(x)t + \dots + P_1(x)P_2(x)t^3 + \dots\right]$$

[first orthoganal Property]

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \int_{-1}^{1} \sum_{n=0}^{\infty} (P_n(x))^2 t^{2n} \cdot dx$$

(using orthogonal property)

$$\int_{-1}^{1} (P_n(x))^2 \, dx = \frac{2}{2n+1}$$

$$\Rightarrow \int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}.$$

(i).

# Unit - IV

## Bessel function:-

The Differential equation is $x^2y'' + xy' + (x^2 - p^2)y = 0$. where p is non-negative its called the Bessal equations. Its solution is known as the Bessal function.

U.Q. Solve Bessal equation:-

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \longrightarrow (1)$$

$$y'' + \frac{y'}{x} + \frac{(x^2 - p^2)}{x^2}y = 0.$$

there $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2 - p^2}{x^2}$

$x = 0$ is a Singular point

there $x_0 = 0$.

$$x\,P(x) = x \cdot 1/x = 1. \Rightarrow x \cdot (1/x)$$

$$x^2 \cdot Q(x) = x^2\left(\frac{x^2 - p^2}{x^2}\right) = x^2 - p^2$$

$x_0$ is a regular Singular point.

The initial equation

$m(m-1) + mp_0 + q_0 = 0 \longrightarrow (2).$

Here $p_0 = 1$ and $q_0 = -p^2$

$m(m-1) + m - p^2 = 0$

$m^2 - m + m - p^2 = 0$

$m^2 - p^2 = 0$

$m^2 = p^2$

$m = \pm p.$

Take $m = \pm p$, then the equation has the soln of the form.

$$y = x^p \cdot \sum_{n=0}^{\infty} a_n x^n \longrightarrow (*)$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+p} \longrightarrow (3).$$

Put $n = n-2$

$$y = \sum_{n=0}^{\infty} a_{n-2} \, x^{n+p-2} \longrightarrow (4)$$

$$y' = \sum_{n=1}^{\infty} (n+p) \, a_n \, x^{n+p-1}$$

$$xy' = \sum_{n=1}^{\infty} (n+p)\, a_n x^{n+p} \longrightarrow (5)$$

$$y'' = \sum_{n=2}^{\infty} (n+p)(n+p-1)\, a_n x^{n+p-2}$$

$$x^2 y'' = \sum_{n=2}^{\infty} (n+p)(n+p-1)\, a_n x^{n+p} \longrightarrow (6).$$

$$x^2 y = x^2 \sum_{n=0}^{\infty} a_{n-2} x^{n+p-2} \quad [\text{by } (4)]$$

$$x^2 y = \sum_{n=0}^{\infty} a_{n-2} x^{n+p} \longrightarrow (7)$$

$$-p^2 y = -p^2 \sum_{n=0}^{\infty} a_n x^{n+p} \longrightarrow (8).$$

Substitude (5), (6), (7), (8) in (1).

$$\sum_{n=2}^{\infty} (n+p)(n+p-1)\, a_n x^{n+p} + \sum_{n=1}^{\infty} (n+p)\, a_n x^{n+p} + \sum_{n=0}^{\infty} a_{n-2} x^{n+p}$$

$$- p^2 \sum_{n=0}^{\infty} a_n x^{n+p} = 0.$$

Equating coefficients of $x^{n+p}$ to zero,

$$\Rightarrow (n+p)(n+p-1)\, a_n + (n+p)\, a_n + a_{n-2}$$

$$-p^2 a_n = 0$$

$$\Rightarrow (n+p)\ (n+p-1)\ a_n + (n+p)\ a_n - p^2 a_n = -a_{n-2}$$

$$\Rightarrow \quad a_n \left[ (n+p)\ (n+p-1) + (n+p) - p^2 \right] = -a_{n-2}$$

$$\Rightarrow \quad a_n \left[ n^2 + np - n + pn + p^2 - p + n + p - p^2 \right] = -a_{n-2}$$

$$\Rightarrow \quad a_n \left[ n^2 + 2pn \right] = -a_{n-2}$$

$$a_n = \frac{-a_{n-2}}{n^2+2pn} \longrightarrow (4)$$

we know that $a_0 \neq 0$.

Take $a_1 = 0$.

put $n=2$, substitude in (4).

$$a_2 = \frac{-a_{2-2}}{2^2+2(2)(p)} = \frac{-a_0}{4+4p}$$

$$a_2 = \frac{-a_0}{2(2+2p)}$$

put $n=3$, substitude in (4).

$$a_3 = \frac{-a_{3-2}}{3^2+2(3)p} = \frac{-a_1}{9+6p}$$

$$= \frac{-a_1}{3(3+2p)} = 0$$

$$\boxed{a_3 = 0}$$

put $n=4$; Substitude in (A)

$$a_4 = \frac{-a_2}{16+8p}$$

$$= \frac{-\left(\frac{-a_0}{2(2+2p)}\right)}{16+8p}$$

$$= \frac{\frac{a_0}{2(2+2p)}}{4(4+4p)}$$

$$a_4 = \frac{a_0}{2.4 \cdot (2+2p)(4+4p)}$$

$$a_5 = a_7 = a_9 = \cdots = 0.$$

put $n=6$,

$$a_6 = \frac{-a_{6-2}}{36+12p} = \frac{-a_4}{6(6+2p)}$$

$$= \frac{-a_0}{2.4.6\,(2+2p)(4+2p)(6+2p)}$$

The soln is,

By $\textcircled{*} \Rightarrow y = x^p [a_0 + a_1 x + a_2 x^2 + \dots]$

$$y = x^p \left[ a_0 - \frac{a_0}{2(2+2p)} \right] x^2$$

$$+ \left[ \frac{a_0}{2.4\ (2+2p)(4+2p)} \right] x^4$$

$$- \left( \frac{a_0}{2.4.6\ (2+2p)(4+2p)(6+2p)} \right) x^6 + \dots$$

$$y = a_0 x^p \left[ 1 - \frac{x^2}{2^2(1+p)} + \frac{x^4}{2^4 . 2!\ (1+p)(2+p)} - \frac{x^6}{2^6 . 3!\ (1+p)(2+p)(3+p)} + \dots \right]$$

$$y = a_0 x^p \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} . n!\ (p+1)(p+2)(p+3) \dots (p+n)}$$

Definition of $J_p(x)$

The Bessal function of the first kind of the order p;

denoted by $J_p(x)$ is defined by putting $a_0 = \frac{1}{2^p . p!}$

$$J_p(x) = \frac{1}{2^p . p!} x^p \sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n}}{2^{2n} . n! \, (p+1)(p+2)(p+3) + \cdots + (p+n)}$$

$$= \frac{1}{2^p} x^p \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} . n! \, (p+n)!}$$

$$= \frac{x^{2n+p}}{2^{2n+p}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, (p+n)!}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n! \, (p+n)!}$$

$$J_p(x) = \frac{\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n! \, (p+n)!} \longrightarrow (1).$$

putting $\underline{p=0}$.

Bessal function of order $p=0$.

$$J_0(x) = \frac{\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n}}{n! \, n!}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n}}{(n!)^2}$$

$$= 1 - \frac{(x/2)^2}{1!} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/4)^6}{(3!)^2} + \cdots$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Bessal function of order $P=1$

$$J_1(x) = \frac{\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n+1}}{n!\ (1+n)!}$$

$$= \frac{x/2}{1!} - \frac{(x/2)^3}{1!\,2!} + \frac{(x/2)^5}{2!\,3!} - \frac{(x/2)^7}{3!\,4!} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{1}{1!\,2!}(x/2)^3 + \frac{1}{2!\,3!}(x/2)^5 - \frac{1}{3!\,4!}\left(\frac{x}{4}\right)^7$$

Gamma functions:-

The Gamma function $\Gamma(p)$ is defined as $\Gamma(P) = \int_0^{\infty} e^{-t}\, t^{P-1}\, dt \longrightarrow (1)$

## Result:-

1.) $\Gamma(p+1) = p\Gamma(p)$

Proof:-

$$\Gamma(p+1) = \int_0^\infty e^{-t} t^{p+1-1} dt$$

$$= \int_0^\infty e^{-t} t^p dt.$$

$$= \left[-t^p e^{-t}\right]_0^\infty + \int_0^\infty e^{-t} t^{p-1} p\, dt$$

$$= 0 + p\int_0^\infty e^{-t} t^{p-1} dt$$

$u = t^p$

$du = p t^{p-1} dt$

$v = e^{-t} dt$

$dv = -e^{-t}$

$$\Gamma(p+1) = p\Gamma(p) \quad [\text{by } ①]$$

2.) prove that, $\Gamma(n+1) = n!$

By above result $(\because \Gamma(p+1) = p\Gamma(p))$

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3)\Gamma(n-3)\ldots$$

$$= n(n-1)(n-2)(n-3)(n-4)\ldots 1\,\Gamma(1)$$

$(\because \Gamma(1) = 1)$

$$\Gamma(n+1) = n!$$

3.) prove that $\Gamma(1) = 1$.

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt$$

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt$$

$$= \int_0^\infty e^{-t} t^0 dt$$

$$= \int_0^\infty e^{-t} dt \qquad \left[\because \int e^{ax} dx = \frac{e^{ax}}{a}\right.$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^\infty$$

$$= \left(-e^{-t}\right)_0^\infty = -e^{-\infty} + e^{-0}$$

$$= 0 + 1 = 0.$$

$$\Gamma(1) = 1.$$

(4) prove that $\Gamma(1/2) = \sqrt{\pi}$

(U.Q)

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt \longrightarrow (1)$$

$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{1/2 - 1} dt$$

$$= \int_0^\infty e^{-t} t^{-1/2} dt$$

put $t=x^2$, $dt = 2x\,dx$.

$t=0 \Rightarrow x=0$.

$t=\infty \Rightarrow x=\infty$.

$$\Gamma(1/2) = \int_0^\infty e^{-x^2} x^{2(-1/2)}\, 2x\,dx$$

$$= \int_0^\infty e^{-x^2} x^{-1}\, 2x\,dx = \int_0^\infty e^{-x^2}\, 2x^{1-1}\,dx = \int_0^\infty e^{-x^2}\cdot 2\,dx$$

$\boxed{x^0=1}$

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}\,dx$$

$$(\Gamma(1/2))^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

$x = r\cos\theta \Rightarrow dy/dr = -r\sin\theta + \cos\theta\cdot 1$

$y = r\sin\theta \Rightarrow dy/dr = r\cos\theta + 1\cdot\sin\theta$

$dx\,dy = r\cdot dr\,d\theta$ $\quad \frac{dx}{dr}\cdot\frac{dy}{dr} = (-r\sin\theta+\cos\theta)(r\cos\theta+\sin\theta)$

$x=0 \Rightarrow r=0$, $x=\infty \Rightarrow r=\infty$. $\quad \frac{dx}{d\theta} = -r\sin\theta$

$y=0 \Rightarrow \theta=0$, $y=\infty \Rightarrow \theta=\pi/2$ $\quad \frac{dy}{d\theta} = r\cos\theta$

$\frac{dy}{dx} = \frac{r}{r}\frac{\cos\theta}{\sin\theta}$

$$(\Gamma(1/2))^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\cdot dr\cdot d\theta$$

$$= 4\int_0^{\pi/2} d\theta \int_0^\infty e^{-r^2}\, r\cdot dr$$

$$= 4[\theta]_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr$$

$$= 4\,(\pi/2) \int_0^\infty e^{-r^2} \cdot r\,dr \qquad = 2\pi \int_0^\infty e^{-r^2} r\,dr$$

$$= 2\pi \cdot \tfrac{1}{2} \int_0^\infty e^{-r^2} \cdot 2r\,dr \qquad = 2\pi \left(\tfrac{2}{2}\right) \int_0^\infty e^{-r^2} r\,dr$$

$$= 2\pi\, \tfrac{1}{2} \int_0^\infty e^{-r^2} \cdot 2r\,dr$$

$$= \pi \int_0^\infty e^{-r^2}\, d(r^2) \qquad = \pi \int_0^\infty e^{-r^2} \cdot 2r\,dr$$

$$= \pi \left[-e^{-r^2}\right]_0^\infty$$

$$= \pi \left[-e^{-\infty} + e^0\right] = \pi$$

$$\left(\Gamma(1/2)\right)^2 = \pi$$

$$\Gamma(1/2) = \sqrt{(\pi)}$$

---

(4)
(a) $$(n+1/2)! = \frac{(2n+1)!}{2^{2n+1}(n!)} \cdot \sqrt{\pi}$$

Soln

W.K.T $\Gamma(n+1) = n!$

$$(n+1/2)! = \Gamma(n+1/2+1)$$

$$= (n+1/2)\,\Gamma(n+1/2)$$

$$= (n+1/2)(n+1/2-1)\,\Gamma(n+1/2-1)$$

$$= (n+1/2)(n+1/2-1)(n+1/2-2)\,\Gamma(n+1/2-2)$$

$\vdots$ $\quad\vdots\quad$ $\vdots$

$$= \left(n+\tfrac{1}{2}\right)\left(n+\tfrac{1}{2}-1\right) \cdots \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \left(\frac{2n+1}{2}\right)\left(\frac{2n-1}{2}\right) \cdots \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi}$$

$$= \left(\frac{2n+1}{2}\right)\left(\frac{2n-1}{2}\right) \cdots \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n+1)(2n-1)\cdots 5\cdot 3\cdot 1}{2\cdot 2\cdots\cdots 2\cdot 2\cdot 2}\sqrt{\pi}$$

$$= \cancel{\frac{(2n+1)(2n)(2n-1)\cdots 5\cdot 3\cdot 1}{(2n+1)(2n)(2n-2)(2n-4)\cdots 4\cdot 2}}\sqrt{\pi}$$

$$= \frac{(2n+1)(2n)(2n-1)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}{(2n+1)(2n)(2n-2)(2n-4)\cdots 4\cdot 2}\sqrt{\pi}$$

$$= \frac{(2n+1)!}{2^{n+1}\cdot 2^{n}(n!)}\sqrt{\pi}$$

$$= \frac{(2n+1)!}{2^{n+1}\cdot(n!)}\sqrt{(\pi)}$$

$$\left(n+\tfrac{1}{2}\right)! = \frac{(2n+1)!}{2^{n+1}\cdot(n!)}\sqrt{\pi}$$

(b.) $(n-1/2)! = \frac{(2n)!}{2^{2n} \cdot n!}\sqrt{\pi}$

<u>Soln.</u>

W.K.T $\Gamma(n+1) = n!$

$(n-1/2)! = \Gamma(n-1/2+1)$

$= (n-1/2)\Gamma(n-1/2)$

$= (n-1/2)(n-3/2)\Gamma(n-3/2)$

$= (n-1/2)(n-3/2)(n-5/2)\Gamma(n-5/2)$

$= (n-1/2)(n-3/2) \cdots 3/2 \cdot 1/2\,\Gamma\pi$

$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right) \cdots 3/2 \cdot 1/2\sqrt{\pi}$

$= \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{2 \cdot 2 \cdots 2 \cdot 2 \cdot 2}\sqrt{\pi}$

$= \frac{(2n)(2n-1)(2n-2)\cdots 4 \cdot 3 \cdot 2}{2^n(2n)(2n-2)(2n-4)\cdots 4 \cdot 2}\sqrt{\pi}$

$= \frac{(2n)!}{2^n \cdot 2^n (n!)}\sqrt{\pi}$

$(n-1/2)! = \frac{(2n)!}{2^{2n} \cdot (n!)}$

(5a.) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

<u>soln:</u>

W.K.T $\quad J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(p+n)!}$

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+1/2}}{n!\,(n+1/2)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1/2}}{n!\, 2^{2n+1/2} \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}}} \quad [\because \text{by } 4(a)]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1/2+1}}{2^{2n+1}\, 2^{-1/2} \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} x^{-1/2}}{2^{-1/2} (2n+1)!\sqrt{\pi}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}\sqrt{2}}{\sqrt{x}\,(2n+1)!\sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sqrt{\frac{2}{\pi x}} \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

4.

b) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Soln:

W.K.T,

$$J_p(x) = \frac{\sum_{n=0}^{\infty} (-1)^n (x/2)^{2n+p}}{n!\,(p+n)!}$$

$$J_{-1/2}(x) = \frac{\sum_{n=0}^{\infty} (-1)^n . (x/2)^{2n-1/2}}{n!\,(n-1/2)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n-1/2}}{2^{2n-1/2}\, n!\,(n-1/2)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n} . x^{-1/2}}{2^{2n} . (2)^{-1/2} \frac{(2n)!}{2^{2n}} \sqrt{\pi}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n}) \sqrt{2}}{\sqrt{x}\, (2n)!\, \sqrt{\pi}}$$

$$= \sqrt{\frac{2}{x\pi}} . \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \sqrt{\frac{2}{x\pi}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{2\pi}} \cos x //.$$

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6 a.) $\frac{d}{dx} J_0(x) = -J_1(x)$

(5.0 marks)

Soln.

w.k.T,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 - \cdots$$

$$\frac{d}{dx} J_0(x) = \frac{d}{dx}\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 6^2} + \cdots\right]$$

$$= \frac{-2x}{2^2} + \frac{4x^3}{2^2 \cdot 4^2} - \frac{6x^5}{2^2 \cdot 4^2 \cdot 6^2}$$

$$\frac{d}{dx} J_0(x) = \frac{-x}{2} + \frac{x^3}{2^3} \cdot \frac{1}{1!\,2!} + \frac{x^5}{2^5} \cdot \frac{1}{2!\,3!} - \cdots$$

$$= -\left[\frac{x}{2} - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 + \cdots\right]$$

$$\frac{d}{dx} J_0(x) = -J_1(x)$$

(b) $\frac{d}{dx}\left[x \cdot J_1(x)\right] = x J_0(x)$

Soln. w.k.T, $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

$$J_1(x) = \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \cdots$$

$$x J_1(x) = x\left[\frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 \cdots\right.$$

$$= \frac{x^2}{2} - \frac{x^4}{2^4} + \frac{x^6}{2^6 \cdot 6} - \cdots$$

$$\frac{d}{dx}\left(x J_1(x)\right) = x - \frac{x^3}{2^2} + \frac{x^5}{2^6} - \cdots$$

$$= x\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^6} - \cdots\right]$$

$$= x J_0(x)$$

$$\frac{d}{dx}\left[x \cdot J_1(x)\right] = x J_0(x)$$

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U.Q

8mark

## Properties of Bessal function

Identifies and the functions

$J_{n+1/2}(x)$.

(i.e), To find Bessal function $J_{n+1/2}(x)$. where $n$ is an integer.

Also find the values of $J_{3/2}$, $J_{5/2}$, $J_{-3/2}$, $J_{-5/2}$

Soln.

We know That,

$$\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$$

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x) \longrightarrow (1).$$

put $p = 1/2$

$$J_{3/2}(x) = \frac{2(1/2)}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= 1/x \, J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x} \sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[ 1/x \sin x - \cos x \right]$$

put $p = 3/2$.

$$J_{5/2}(x) = \frac{2(3/2)}{x} J_{3/2}(x) - J_{1/2}(x)$$

$$= 3/x \left[ \sqrt{\frac{2}{\pi x}} \left( 1/x \sin x - \cos x \right) \right] - \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[ \frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x \right]$$

Similarity if we put $P = 3/2, 1/2$,

we get $J_{-1/2}, J_{9/2}$

From (1),

$$J_{p-1}(x) = \frac{2p}{x} J_p(x) - J_{p+1}(x) \longrightarrow (2).$$

Put $P = -1/2$

$$J_{-3/2}(x) = \frac{-2(1/2)}{x} J_{-1/2}(x) - J_{(-1/2+1)}(x)$$

$$= -\frac{1}{x} J_{-1/2} x - J_{1/2}(x)$$

$$= \frac{-1}{x}\sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}\{\cos x + \sin x\right]$$

put $P = -3/2$

$$J_{-\frac{5}{2}}(x) = \frac{2(-3/2)}{x} J_{-\frac{3}{2}}(x) - J_{\left(-\frac{3}{2}+1\right)}(x)$$

$$= \frac{-3}{x} J_{-\frac{3}{2}}(x) - J_{-1/2}(x)$$

$$= \frac{-3}{x}\left[\frac{-2}{\pi x}\left(\frac{1}{x}\cos x + \sin x\right)\right] - \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right]$$

Similarly we put $p = -5/2, -7/2, \ldots$

we get $J_{-\frac{7}{2}}, J_{-\frac{9}{2}},$

Hence every Bessals function $J_{n+\frac{1}{2}}(x)$ where $n$ is any integer can be determined.

---

1.) prove that

IQ. (a) $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p \cdot J_{p-1}(x)$

(b) $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} \cdot J_{p+1}(x).$

Soln.

w.k.T $J_p(x) = \dfrac{\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(p+n)!}$

$$x^p J_p(x) = x^p \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n+p} \cdot n!\,(p+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{2^{2n+p} \cdot n!\,(p+n)!}$$

$$\frac{d}{dx}\left[x^p J_p(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p)\, x^{2n+2p-1}}{2^{2n+1}\, n!\,(p+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2(n+p)\, x^{2n+2p-1}}{2^{2n+p}\, n!\,(p+1)(p+n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2p-1}}{2^{2n+p-1}\, n!\,(p+n-1)!}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+p-1}}{2^{2n+p-1}\, n!\,(p+n-1)!}$$

$$= x^p J_{p-1}(x)$$

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p.\, J_{p-1}(x)$$

(b). $$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

Soln.

w.k.T, $$J_p(x) = \frac{\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(p+n)!}$$

$$x^{-p} J_p(x) = \frac{x^{-p} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(p+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+p}\, n!\,(p+n)!}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n \, 2n \, . \, x^{2n-1}}{2^{2n+p} \, n(n-1)! \, (p+n)!}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n \, x^{2n-1}}{2^{2n+p-1} (n-1)! \, (p+n)!}$$

$$= x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+p-1}}{2^{2n+p-1} (n-1)! \, (p+n)!}$$

$$= -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+p-1}}{2^{2n+p-1} (n-1)! \, (p+n-1+1)!}$$

$$= \frac{-x^{-p} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n+p-1+1-1}}{(n-1)! \, (n-1+p+1)!}$$

$$= -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p+1}}{n! \, (n+p+1)!}$$

(replacing $(n-1)$ by $n$)

$$= -x^{-p} J_{p+1}(x)$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x).$$

//.

2

Prove That

(a) $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$

(b) $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

Hence derived

$$J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$

Soln.

W.K.T $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$

$$x^p \cdot J_p(x) + J_p(x)\, p x^{p-1} = x^p J_{p-1}(x) \longrightarrow (1)$$

W.K.T, $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

$$x^{-p} J_p'(x) + J_p(x)(-p)\, x^{-p-1} = -x^{-p} J_{p+1}(x).$$

$$x^{-p} J_p'(x) - J_p(x)\, p\, x^{-p-1} = -x^{-p} J_{p+1}(x) \longrightarrow (2)$$

(1) $\div x^p$

$$J_p(x) + J_p(x)\, p x^{-1} = J_{p-1}(x) \longrightarrow (3)$$

(2) $\div x^{-p}$

$$J_p'(x) - J_p(x)\, p x^{-1} = -J_{p+1}(x) \longrightarrow (4)$$

(3) + (4)

$$2 J_p'(x) = J_{p-1}(x) - J_{p+1}(x) \longrightarrow (5)$$

(3) - (4) $\Rightarrow$

$$2 J_p'(x) p x^{-1} = J_{p-1}(x) + J_{p+1}(x)$$

(i.e) $$2 J_p(x) \frac{p}{x} = J_{p-1}(x) + J_{(p+1)}(x) \rightarrow (6)$$

(5) + (6)

$$2 J_p'(x) + 2 J_p(x) \frac{p}{x} = 2 J_{p-1}(x).$$

$$2\left[J_p'(x) + \frac{p}{x} J_p(x)\right] = 2 J_{p-1}(x)$$

$$J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$ //.

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(3)

## Orthoganal property of Bessal functions.

prove that

$$\int_0^1 x J_p(\lambda_m(x)) J_p(\lambda_n(x)) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{2} J_{p+1}(\lambda_n)^2 & \text{if } m = n \end{cases}$$

**proof.**

$y = J_p(x)$ is a soln of a Bessal equation

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

$$y'' + \frac{y'}{x} + \left(1 - \frac{p^2}{x^2}\right) y = 0 \longrightarrow (1)$$

If $a$ and $b$ are distinct constant it follows that the function

$u(x) = J_p(ax)$ and

$v(x) = J_p(bx)$ satistifies (1)

$$u'' + \frac{1}{x} u' + \left(a^2 - \frac{p^2}{x^2}\right) u = 0 \longrightarrow (2)$$

$$v'' + \frac{1}{x} v' + \left(b^2 - \frac{p^2}{x^2}\right) v = 0 \longrightarrow (3)$$

$$(2) \times v \Rightarrow u''v + \frac{1}{x} u'v + \left(a^2 - \frac{p^2}{x^2}\right) uv = 0 \longrightarrow (4)$$

$$(3) \times u \Rightarrow uv'' + \frac{1}{x} uv' + \left(b^2 - \frac{p^2}{x^2}\right) uv = 0 \longrightarrow (5)$$

(4) - (5)

$$\Rightarrow u''v + \frac{1}{x} u'v + a^2uv - \cancel{\frac{p^2}{x^2}} uv$$

$$- v''u - \frac{1}{x} v'u - b^2uv + \cancel{\frac{p^2}{x^2}} uv = 0$$

$$(u''v - v''u) + \frac{1}{x}(u'v - v'u) + uv(a^2 - b^2) = 0$$

$$\frac{d}{dx}(u'v - v'u) + \frac{1}{x}(u'v - v'u) + uv(a^2 - b^2) = 0$$

$$x \frac{d}{dx}(u'v - v'u) + u'v - v'u - x(b^2 - a^2)uv = 0.$$

$$\frac{d}{dx} x(u'v - v'u) + (u'v - v'u) = x(b^2 - a^2)uv$$

$$\frac{d}{dx}\left[x(u'v - v'u)\right] = x(b^2 - a^2)uv$$

Integrating with respect to $x$ from 0 to 1.

$$\left[x(u'v - v'u)\right]_0^1 = (b^2 - a^2)\int_0^1 x\,uv\,dx.$$

The expression in brackets vanish at $x = 0$.

$u(x) = J_p(ax)$, $v(x) = J_p(bx)$

$u(1) = J_p(a)$, $v(1) = J_p(b)$

$\therefore$ The integral part is 0 if $a$ and $b$ are distinct positive zero of $\lambda_m$ and $\lambda_n$ of $J_p(x)$.

$$\int_0^1 x J_p(\lambda_m(x)) J_p(\lambda_n x)\,dx = 0 \quad \text{if } m \neq n.$$

when $m = n$

(2) $\times\, 2x^2u' \Rightarrow u''(2x^2u') + \frac{1}{x}u'(2x^2u') + \left(a^2 - \frac{p^2}{x^2}\right)u(2x^2u') = 0$

$$\Rightarrow 2x^2u'u'' + 2x(u')^2 + \left(\frac{a^2x^2-p^2}{x^2}\right)(2x^2uu') = 0.$$

$$\Rightarrow 2x^2u'u'' + 2x(u')^2 + (a^2x^2-p^2)2uu' = 0$$

$$2x^2u'u'' + 2x(u')^2 + a^2x^2 2uu' - 2p^2uu'$$

$$+ 2a^2xu^2 - 2a^2xu^2 = 0.$$

$$\frac{d}{dx}\left(x^2(u')^2\right) + \frac{d}{dx}(a^2x^2u^2) - 2a^2xu^2$$

$$-\frac{d}{dx}(p^2u^2) = 0.$$

$$\left(\because \frac{d}{dx}(a^2x^2u^2) = a^2x^2(2u)u' + a^2 2x.u^2\right)$$

$$\frac{d}{dx}\left[x^2(u')^2 + a^2x^2u^2 - p^2u^2\right] = 2a^2xu^2$$

Integrating from 0 to 1.

$$\left[x^2(u')^2 + a^2x^2u^2 - p^2u^2\right]_0^1 = 2a^2\int_0^1 xu^2\,dx \quad \text{— (5)}$$

$$u(x) = J_p(ax) \Rightarrow u'(x) = J_p'(ax)\,a$$

$$u(1) = J_p(a) \Rightarrow u'(1) = J_p'(a)\,a$$

$$(5) \Rightarrow (1)^2(u'(1))^2 + a^2(1)^2(u(1))^2 - p^2(u(1))^2$$

$$-(0) + (0) - p^2(u(0))^2] = 2a^2\int_0^1 xu^2\,dx$$

$$\left[u'(1)^2 + a^2 (u(1))^2 - p^2 (u(1))^2 - p^2(0)\right] = 2a^2 \int_0^1 x u^2 \, dx$$

$$\int_0^1 x u^2 \, dx = \frac{1}{2a^2} a^2 J_p'(a)^2 + \frac{a^2}{2a^2} J_p(a^2) - \frac{p^2}{2a^2} J_p(a)^2 = \frac{1}{2} J_p'(a)^2 + \frac{1}{2} J_p(a^2) - \frac{p^2}{2a^2} J_p(a)^2$$

$$\int_0^1 x J_p(ax)^2 \, dx = \frac{1}{2} J_p'(a)^2 + \frac{1}{2} J_p(a)^2 \left(1 - \frac{p^2}{a^2}\right)$$

put $a = \lambda n$ we get

$$\int_0^1 x J_p(\lambda_n x)^2 \, dx = \frac{1}{2} J_p'(\lambda n)^2 + J_p(\lambda n)^2 \left(1 - \frac{p^2}{(\lambda n)^2}\right)$$

$$J_p'(x) - \frac{p}{x} J_p(x) = J_{p+1}(x) \qquad (\text{put } x = (\lambda n)^2)$$

$$J_p'(\lambda n)^2 - \frac{p}{(\lambda n)^2} \cdot J_p(\lambda n)^2 = J_{p+1}(\lambda n)^2$$

$$J_p'(\lambda n)^2 = J_{p+1}(\lambda n^2) \longrightarrow (7)$$

$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$

$\sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{x}{2}\right)^{2n+p} / n!\,(p+n)!$

$(\because \lambda n$ is zero of Bessal function$)$

$$\int_0^1 x J_p'(\lambda n x)^2 \, dx = \frac{1}{2} J_p'(\lambda n)^2 \quad [\text{by } (7)]$$

$$= \frac{1}{2} J_{p+1}(\lambda n)^2 \quad \text{if } m = n$$

$$\int_0 x\, J_p(\lambda_n x)\, J_p(\lambda_m x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{2} J_{p+1}(\lambda_m)^2 & \text{if } m = n \end{cases}$$

Imp U.Q

**Problem.**

Express $J_2(x)$, $J_3(x)$ and $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$

**Soln:**

W.k.T,

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$$

put $p=1$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

put $p=2$,

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$= \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$= \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

$$J_3(x) = J_1(x)\left[\frac{8}{x^2} - 1\right] - \frac{4}{x} J_0(x)$$

put $p=3$

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left[J_1(x)\left(\frac{8}{x^2}-1\right) - \frac{4}{x} J_0(x)\right]$$

$$-\left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \frac{48}{x^3} J_1(x) - \frac{6}{x} J_1(x) - \frac{24}{x} J_0(x)$$

$$-2/x\, J_1(x) - J_0(x)$$

$$J_4(x) = J_1(x)\left[\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right] + J_0(x)\left[1 - \frac{24}{x^2}\right]$$

$$J_4(x) = J_1(x)\left[\frac{48}{x^3} - \frac{8}{x}\right] + J_0(x)\left[1 - \frac{24}{x^2}\right]$$ //

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(5)

**Problem:**

U.Q If $f(x)$ is defined by

$$f(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ 1/2 & x = 1/2 \\ 0 & 1/2 < x \le 1 \end{cases}$$

Such that $f(x) = \sum_{n=1}^{\infty} \frac{J_1(\lambda n/2)}{\lambda n J_1(\lambda n)^2} J_0(\lambda n(x))$

**Soln:**

The Bessal Series function is

given by $f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n(x))$

where,

$$a_n = \frac{2}{J_{p+1}(\lambda_n)^2} \int_0^1 x f(x) J_p(\lambda_n(x))\, dx$$

put $p = 0$.

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n(x)) \longrightarrow (1).$$

where $a_n = \dfrac{2}{J_1(\lambda_n)^2} \displaystyle\int_0^1 x f(x) J_0(\lambda_n x)\, dx$

$$a_n = \frac{2}{J_1(\lambda_n)^2} \left[ \int_0^{1/2} x J_0(\lambda_n(x))\, dx + \int_{1/2}^1 0\, dx \right]$$

$$= \frac{2}{J_1(\lambda_n)^2} \int_0^{1/2} x J_0 \lambda_n(x)\, dx.$$

$$= \frac{2}{J_1(\lambda_n)^2} \left[ \frac{1}{\lambda_n} x J_1(\lambda_n(x)) \right]_0^{1/2}$$

$$= \frac{2}{J_1(\lambda_n)^2 \cdot \lambda_n} \left[ x J_1 \lambda_n(x) \right]_0^{1/2}$$

$$= \frac{2}{J_1(\lambda_n)^2 \cdot \lambda_n} \left[ \tfrac{1}{2} J_1 \cdot \lambda_n(\tfrac{1}{2}) \right]$$

$$= \frac{1}{\lambda_n J_1(\lambda_n)^2} J_1(\lambda_n/2)$$

$$a_n = \frac{J_1(\lambda n/2)}{\lambda n\, J_1(\lambda n)^2}$$

① $\Rightarrow$ $$f(x) = \sum_{n=1}^{\infty} \frac{J_1(\lambda n/2)}{\lambda n\,(J_1)(\lambda n)^2} J_0(\lambda n x)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{J_1(\lambda n/2)}{\lambda n\, J_1(\lambda n)^2} J_0(\lambda n(x))$$

Unit-V

Remark (i) Procedure to solve non-homogenous linear system

Consider the non-homogenous linear system

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t)\\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t)\end{aligned}\right\} \rightarrow (1)$$

If $f_1(t)$ and $f_2(t)$ are identically zero.

Then the system (1) is called homogenous. otherwise it is said to be non-homogenous.

The corresponding homogenous system is

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y \text{ and}$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y$$

If the homogeneous system

has two solutions.

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \text{ and } \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases} \text{ on } [a,b]$$

simultaneously

Then $\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$ is also

a solution on $[a,b]$ for any

constants $c_1$ and $c_2$

now $x = v_1(t)\, x_1(t) + v_2(t)\, x_2(t)$ and

$y = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$

is a particular solution of (1).

If the function $v_1(t)$ and

$v_2(t)$ satisfies the system

$v_1'(x_1) + v_2'(x_2) = f_1$

$v_1'(y_1) + v_2'(y_2) = f_2$

Theorem: A

If $t_0$ is any Point of the $[a,b]$ and if $x_0$ and $y_0$ are any numbers what ever, then

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(x)y + f_1(t) \\ \frac{dy}{dt} &= a_2(t)x + b_2(x)y + f_2(t) \end{aligned}\right\} \text{ has}$$

one and only Solution $\left\{\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}\right\}$

valid throughout $[a,b]$, such that

$x(t_0) = x_0$ and $y(t_0) = y_0$.

Theorem: B

U.Q

If the homogenous system

$$\left\{\begin{aligned} \frac{dx}{dt} &= a_1(t) + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right. \quad \text{has two soln}$$

$$\left\{\begin{aligned} x &= x_1(t) \\ y &= y_1(t) \end{aligned}\right. \text{ and } \left\{\begin{aligned} x &= x_2(t) \\ y &= y_2(t) \end{aligned}\right. \text{ on } [a,b]$$

Then

$$\begin{cases} x = c_1 x_1(t) + c_2 (x_2)t \\ y = c_1 (y_1)t + c_2 (y_2)t \end{cases}$$ is also a

Solution on $[a,b]$ for any Constants $c_1$ and $c_2$

<u>proof:-</u>

Given the homogenous System

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \rightarrow (1)$$

having the two Solutions are,

$$\left.\begin{aligned} x &= x_1(t) \\ y &= y_1(t) \end{aligned}\right\} \text{ and } \left.\begin{aligned} x &= x_2(t) \\ y &= y_2(t) \end{aligned}\right\} \text{ on } [a,b]$$

Since $x_1(t)$ and $y_1(t)$ are Solutions of (1).

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_1(t)x_1 + b_1(t)y_1 \\ \frac{dy_1}{dt} &= a_2(t)x_1 + b_2(t)y_1 \end{aligned}\right\} \rightarrow (3)$$

Since $x_2(t)$ and $y_2(t)$ are

Solutions of (1).

$$\left.\begin{aligned} \frac{dx_2}{dt} &= a_1(t)x_2 + b_1(t)y_1 \\ \frac{dy_2}{dt} &= a_2(t)x_2 + b_2(t)y_2 \end{aligned}\right\} \rightarrow (3)$$

Equation (2) is multiple by

$c_1$ and eqn (3) multiple by $c_2$.

$$\left.\begin{aligned} (2)\times c_1 \Rightarrow c_1 \cdot \frac{dx_1}{dt} &= c_1 a_1 x_1 + c_1 b_1 y_1 \\ c_1 \cdot \frac{dy_1}{dt} &= c_1 a_2 x_1 + c_2 b_2 y_1 \end{aligned}\right\} \rightarrow (4)$$

$$\left.\begin{aligned} (3)\times c_2 \Rightarrow c_2 \cdot \frac{dx_2}{dt} &= c_2 a_1 x_2 + c_2 b_1 y_2 \\ c_2 \cdot \frac{dy_2}{dt} &= c_2 a_2 x_2 + c_2 b_2 y_2 \end{aligned}\right\} \rightarrow (5)$$

then adding (4) and (5)

$$c_1 \frac{dx_1}{dt} + c_2 \cdot \frac{dx_2}{dt} = c_1 a_1 x_1 + c_1 b_1 y_1 + c_2 a_1 x_2 + c_2 b_1 y_2$$

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$$\frac{d}{dt}(c_1x_1+c_2x_2) = a_1(x_1c_1+x_2c_2)$$
$$+b_1(c_1y_1+c_2y_2) \rightarrow (6)$$

Similarly,

$$\frac{d}{dt}(c_1y_1+c_2y_2) = a_1(y_1c_1+y_2c_2)$$
$$+b_1(c_1x_1+c_2x_2) \rightarrow (7)$$

from (6) and (7),

$$\begin{cases} x = c_1x_1(t)+c_2x_2(t) \\ y = c_1y_1(t)+c_2y_2(t) \end{cases}$$

Theorem: C

If the two solutions

$x = x_1(t)$ and $x = x_2(t)$ of the
$y = y_1(t)$ $\quad$ $y = y_2(t)$

homogenous system

$$\begin{cases} \frac{dx}{dt} = a_1(t)x+b_1(t)y \\ \frac{dy}{dt} = a_2(t)x+b_2(t)y \end{cases}$$

have a wronskian $w(t)$ that does not vanish on $[a,b]$. then

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$$ is the general

solutions of homogenous System.

wronskian for general solu

Proof:-

Given the homogenous system

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$$

have a wronskian does not vanish $[a,b]$

i.e) $w(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \neq 0$

$\therefore$ $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ are linearly independent wronskian

Then by Theorem B,

$x = c_1 x_1(t) + c_2 x_2(t)$

is the

$y = c_1 y_1(t) + c_2 y_2(t)$

general solutions of the homogenous system.

**Theorem: D**

If $w(t)$ is the wranskian of the two solutions

$$\left.\begin{array}{l} x = x_1(t) \\ y = y_1(t) \end{array}\right\} - (1) \text{ and } \begin{array}{l} x = x_2(t) \\ y = y_2(t) \end{array} \text{ of}$$

the homogenous system.

$$\left.\begin{array}{l} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{array}\right\} \rightarrow (2)$$

then $w(t)$ is either identically zero or no where zero on $[a,b]$.

**Proof:-**

The wronskian of two solutions of (1) is

$$w(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$w' = x_1 y_2' + x_1' y_2 - x_2 y_1' - x_2' y_1$$

Since equation ① is solution of homogenous system ②

$$\frac{dx_1}{dt} = a_1(t) x_1 + b_1(t) y_1 \rightarrow ③ \text{ and}$$

$$\frac{dy_1}{dt} = a_2(t) x_1 + b_2(t) y_1 \rightarrow ④$$

$$\frac{dx_2}{dt} = a_1(t) x_2 + b_1(t) y_2 \rightarrow ⑤$$

$$\frac{dy_2}{dt} = a_2(t) x_2 + b_2(t) y_2 \rightarrow ⑥$$

$$③ \times y_2 \Rightarrow y_2 \cdot \frac{dx_1}{dt} = a_1 x_1 y_2 + b_1 y_1 y_2 \rightarrow ⑦$$

$$④ \times x_2 \Rightarrow x_2 \cdot \frac{dy_1}{dt} = a_2 x_1 x_2 + b_2 y_1 x_2 \rightarrow ⑧$$

$$⑤ \times y_1 \Rightarrow y_1 \cdot \frac{dx_2}{dt} = a_1 x_2 y_1 + b_1 y_2 y_1 \rightarrow ⑨$$

$$⑥ \times x_1 \Rightarrow x_1 \cdot \frac{dy_2}{dt} = a_2 x_2 x_1 + b_2 y_2 x_1 \rightarrow ⑩$$

(16) + (7)

$$x_1 \cdot \frac{dy_2}{dt} + y_2 \frac{dx_1}{dt} = a_2 x_2 x_1 + b_2 y_2 x_1 + a_1 x_1 y_2 + b_1 y_1 y_2$$

$$x_1 \frac{dy_2}{dt} + y_2 \cdot \frac{dx_1}{dt} - x_2 \frac{dy_1}{dt} - y_1 \frac{dx_2}{dt}$$

$$= a_2 x_2 x_1 + b_2 x_1 x_2 + a_1 x_1 y_2 + b_1 y_1 y_2 - a_2 x_1 x_2 - b_2 y_1 x_2 - a_1 x_2 y_1 + b_1 y_2 y_1$$

$$\frac{d}{dt}(x_1 y_2 - x_2 y_1) = b_2 x_1 x_2 + a_1 x_1 y_2 - a_1 x_2 x_1 - b_1 y_2 y_1$$

$$= x_1 y_2 (b_2 + a_1) - y_1 x_2 (b_2 + a_1)$$

$$\frac{d}{dt}(x_1 y_2 - x_2 y_1) = x_1 y_2 (b_2 + a_1) - x_2 y_1 (b_2 + a_1)$$

$$\frac{d\omega}{dt} = (b_2 + a_1)(x_1 y_2 - x_2 y_1)$$

$$\frac{d\omega}{dt} = (b_2 + a_1)(\omega)$$

$$\frac{dw}{w} = (b_2+a_1)\,dt$$

$$\int \frac{dw}{w} = \int (b_2+a_1)\,dt + c$$

$$\log w = \int (b_2+a_1)\,dt + c$$

$$w = e^{\int (b_2+a_1)\,dt} + e^{c}$$

$$w = c \cdot e^{\int (b_2+a_1)\,dt}$$

W.K.T,

the exponential function is never zero

If $c=0$ is wronskian is zero

If $c \neq 0$ is wronskian is not zero

Hence proved.

Theorem: 5

Q.A

If the two solutions are

$$\left.\begin{array}{ll} x = x_1(t) \text{ and } & x = x_2(t) \\ y = y_1(t) & y = y_2(t) \end{array}\right\} \rightarrow (1)$$

of the homogenous system

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\}$$ are linearly

independent on $[a,b]$. Then

$$\left.\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ y &= c_1 y_1(t) + c_2 y_2(t) \end{aligned}\right\}$$ is the general

solutions of the homogenous System

on this interval.

Proof:

Lemma:

The two solutions of ① are linearly dependent iff their wronskian $w(t)$ is identically zero.

Proof of Lemma:-

Assume that the two solution are linearly dependent

To Prove: $w(t) = 0$

$$x_1(t) = k\, x_2(t)$$
$$y_1(t) = k\, y_2(t)$$

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$$

$$= \begin{vmatrix} k x_2(t) & x_2(t) \\ k y_2(t) & y_2(t) \end{vmatrix}$$

$$= k x_2(t) y_2(t) - x_2(t) k y_2(t) = 0$$

$\therefore W(t) = 0$

Conversely,

Suppose $W(t)$ is identically zero

i.e) $W(t) = 0$

$$W(x_1, x_2) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}$$

$$= x_1 x_2' - x_2 x_1'$$

Since $W = 0$, $x_1 x_2' - x_2 x_1' = 0$

$$\frac{x_1 x_2' - x_2 x_1'}{x_1^2} = 0$$

$$d\left(\frac{x_1}{x_2}\right) = 0, \quad \frac{x_1}{x_2} = k$$

$\therefore x_1$ and $x_2$ are linearly dependent

By theorem D and lemma,

Hence the lemma

Theorem: F

If the two solutions

$$\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix} \quad \text{and} \quad \begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$$ of the

homogenous system,

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \text{ are}$$

linearly independent on $[a,b]$ and

if $\left.\begin{matrix} x = x_p(t) \\ y = y_p(t) \end{matrix}\right\}$ is any Particular

solutions of non-homogenous system

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t) + b_1(t) + f_1(t) \\ \frac{dy}{dt} &= a_2(t) + b_2(t) + f_2(t) \end{aligned}\right\} \text{ Then}$$

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$$\left.\begin{array}{l} x = c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ y = c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{array}\right\}$$ is a

general solution of non-homogenous system.

$$\frac{dx}{dt} = a_1(t) + b_1(t) y + f_1(t)$$

$$\frac{dy}{dt} = a_2(t) + b_2(t) y + f_2(t)$$

proof:

Since (1) is a soln of (2)

$$\left\{\begin{array}{l} \dfrac{dx_1}{dt} = a_1(t) x + b_1(t) y_1 \longrightarrow (6) \text{ and} \\ \dfrac{dy_1}{dt} = a_2(t) x_1 + b_2(t) y_1 \longrightarrow (7) \end{array}\right.$$

$$\left\{\begin{array}{l} \dfrac{dx_2}{dt} = a_1(t) x_2 + b_1(t) y_2 \longrightarrow (8) \text{ and} \\ \dfrac{dy_2}{dt} = a_2(t) x_2 + b_2(t) y_2 \longrightarrow (9) \end{array}\right.$$

Since (3) is a soln of (4)

$$\left\{\begin{array}{l} \dfrac{dx_p}{dt} = a_1(t) x_p + b_1(t) y_p + f_1(t) \longrightarrow (10) \\ \dfrac{dy_p}{dt} = a_2(t) x_p + b_2(t) y_p + f_2(t) \longrightarrow (11) \end{array}\right.$$

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$c_1 \times ① \Rightarrow c_1 \dfrac{dx_1}{dt} = c_1 a_1 x_1 + c_1 b_1 y_1 \rightarrow ⑫$

$c_1 \times ② \Rightarrow c_1 \dfrac{dy_1}{dt} = c_2 a_2 x_1 + c_2 b_2 y_1 \rightarrow ⑬$

$c_2 \times ⑧ \Rightarrow c_2 \cdot \dfrac{dx_2}{dt} = c_2 a_1 x_2 + c_2 b_1 y_2 \rightarrow ⑭$

$c_2 \times ⑨ \Rightarrow c_2 \cdot \dfrac{dy_2}{dt} = c_2 a_2 x_2 + c_2 b_2 y_2 \rightarrow ⑮$

⑫ + ⑭ + ⑩

$$\frac{d}{dt}(c_1 x_1 + c_2 x_2 + x_p) = c_1 a_1 x + c_1 b_1 y_1 + c_2 a_1 x_2 + c_2 b_1 y_2 + a_1 x_p + b_1 y_p + f_1(t) \rightarrow ⑯$$

⑬ + ⑮ + ⑪

$$\frac{d}{dt}(c_1 y_1 + c_2 y_2 + y_p) = c_1 a_2 x_1 + c_1 b_2 y_1 + c_2 a_2 x_2 + c_2 b_2 y_2 + a_2 x_p + b_2 y_p + f_2(t) \rightarrow ⑰$$

(16) $\Rightarrow \frac{d}{dt}(c_1x_1+c_2x_2+x_p) = a_1(c_1x_1+c_2x_2+x_p)$

$+b_1(c_1y_1+c_2y_2+y_p)+f_1(t)$

(17) $\Rightarrow \frac{d}{dt}(c_1y_1+c_2y_2+y_p) = a_2(c_1x_1+c_2x_2+x_p)$

$+b_2(c_1y_1+c_2y_2+y_p)$

$-f_2(t)$

then

$$\left.\begin{aligned} x &= c_1x_1(t)+c_2x_2(t)+x_p(t) \\ y &= c_1y_1(t)+c_2y_2(t)+y_p(t) \end{aligned}\right\} \text{ is}$$

general soln of non homogenous

system

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x+b_1(t)y+f_1(t) \\ \frac{dy}{dt} &= a_2(t)x+b_2(t)y+f_2(t) \end{aligned}\right\}$$

Hence the proof.